NEW
PLANE GEOMETRY

BY

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PREFACE

In the preparation of this text the author acknowledges joint authorship with Robert L. Short.

Geometry is approached from the constructive side, all methods of construction needed for drawing any figure in Books I or II being given in the introduction. In cases where a geometric principle is used in any construction, a note at the end tells where the principle is proved. In all figures in the early portions of Book I, the construction lines and arcs are given; afterwards these are dispensed with.

In Props. II and III, Book I, and in other places, colored diagrams are given in addition to the regular figures, in which the equal parts in the given triangles are represented by *lines of the same color;* this scheme will be found of great assistance to the pupil in the earlier portions of the work.

Below each figure is a paragraph in smaller type, giving full directions for the construction of the diagram in accordance with the statement of the theorem. This gives the pupil a familiarity with the figure, and what in it is given, and what to be proved, that is of great value, and lessens the tendency to memorize. Many figures are omitted, but complete directions for their construction are given in each case.

In all figures given in connection with theorems and problems, given and required lines are made *heavy.*

Attention is invited to the order of theorems in Book I; in this case, the pupil begins with the easier proofs. From the start the student has practice in representing angles and lines by small letters.

Only the outline of the proof is given after Book II, except in the more difficult demonstrations. In this outline work the pupil has explicit directions, but develops the demonstration himself. In all portions of the work proofs are omitted

cases where they should present no difficulty; usually, in such cases, hints are given as to the method of demonstration.

The numbering of the steps of the proof makes them more easy of reference.

In Book I, and in the first part of Book VII, the authority for each statement of the proof will be found directly below, in smaller type, enclosed in brackets, with the number of the section where it is to be found. In other portions of the work only the section number is given, and in some cases only an interrogation point. In all such cases the pupil should be required to give the authority as fully as if it were actually printed on the page.

No principles are given as immediate consequences of theorems except such as actually belong in this category. Separate propositions are made for all truths which are not immediate consequences of preceding theorems.

The originals are new, and very largely of a practical nature. They are not too difficult, will make the pupil think, and are more than five hundred in number. The smaller number is compensated for by the fact that the pupil has to do some original work in almost every proof after Book II. Exercises coming under Book I will be found scattered through all the following Books of the Plane Geometry; and a similar remark applies to exercises under Book II, Book III, etc.

At the end of Book I will be found a list of principles proved, which will be of great assistance in solving originals. A similar list in regard to similar triangles is given in § 265.

The authors wish to thank the many teachers whose advice and criticism have been useful in preparing a treatise that should stand the test of class-room work. They are also indebted to Mr. C. W. Sutton for a part of the exercises.

WEBSTER WELLS.

BOSTON, 1908.

SUGGESTIONS TO TEACHERS

In studying the opening propositions, in Geometry, beginners have difficulty in fixing clearly in mind just what parts, in the figure, are given. To aid this, in Props. II and III, Book I, and in other places, diagrams are given in addition to the regular cuts, in which the equal given parts are printed in the same color.

This color-scheme may be advantageously followed in the class-room, in connection with all figures in the earlier portions of Book I. If colored crayons are not available, the beginner may designate equal lines by the marks ′, ″, ‴, etc., and equal angles by single, double, or triple *arcs*.

In solving the non-numerical exercises, while it is not practicable to give very much assistance to the pupil, the following suggestions may be found of service: —

1. *Draw an accurate figure, showing all given and required lines.*

This sometimes suggests the method of proof.

2. *Be sure that the figure is the most general one allowable.*

Thus, in exercises relating to triangles, do not draw them right, isosceles, or equilateral, unless the exercise calls for a right, isosceles, or equilateral triangle; in exercises relating to quadrilaterals, do not draw them with two sides equal or parallel, unless the exercise calls for such a construction.

3. *Write down carefully what is given, and what is to be proved.*

4. *Look up all previous theorems which may possibly have a bearing on what is to be proved.*

Thus, if two lines are to be proved equal, refer to all previous theorems regarding equal lines. (Compare § 141.)

In solving exercises in construction, it is advantageous to regard the problem as solved, and draw the given and required lines. Studying the relations between these will frequently suggest the method of construction to be employed.

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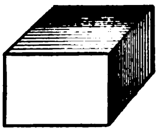
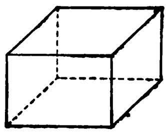
GEOMETRY

INTRODUCTORY



A material body



A geometrical solid

1. A *material body*, as, for example, a block of wood, occupies a *limited* portion of space.

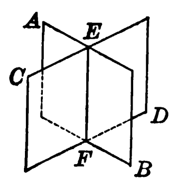
The boundary which separates such a body from surrounding space is called the *surface* of the body.

If the material which composes such a body could be conceived as taken away from it, *without altering the form of the bounding surface*, we should have a *portion of space*, with the same bounding surface as the material body.

We call this portion of space a *geometrical solid*, or simply a *solid*.

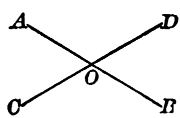
We call the surface which bounds it a *geometrical surface*, or simply a *surface;* it is also called the *surface of the solid.*

2. If two surfaces intersect each other, we call that which is common to both a *geometrical line*, or simply a *line*.



Thus, if surfaces AB and CD cut each other, their common part, EF, is a line.

3. If two lines intersect each other, we call that which is common to both a *geometrical point*, or simply a *point*.



Thus, if lines AB and CD cut each other, their common part, O, is a point.

4. A solid has *extension in every direction;* but this is not the case with surfaces and lines.

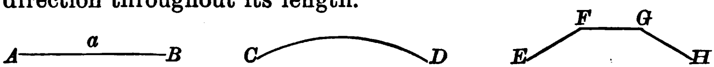
A point has extension in *no* direction.

We may conceive a surface as existing independently in space, without reference to the solid whose boundary it forms.

In like manner, we may conceive of lines and points as existing independently in space.

A line is produced by the motion of a moving point, a surface by the motion of a moving line, and a solid by the motion of a moving surface.

5. We define a *straight line* as a line which has the same direction throughout its length.



A straight line A curve A broken line

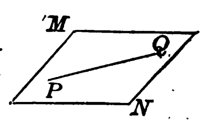
A straight line is designated by two letters anywhere upon its length, or by a single letter; thus, the straight line in the figure may be designated either AB or a.

We define a *curve* as a line no portion of which is straight; as CD.

We define a *broken line* as a line which is composed of different successive straight lines; as $EFGH$.

The word "*line*," without qualification, will be used hereafter as signifying a *straight line.*

6. We define a *plane* as a surface such that if a straight line be drawn between any two of its points, it lies entirely in the surface.



Thus, if P and Q are any two points in surface MN, and the straight line PQ lies entirely in the surface, then MN is a plane.

7. We may conceive a straight line as being of unlimited length; we may also conceive a plane as being of unlimited extent in regard to length and breadth.

8. We define a *geometrical figure* as any combination of points, lines, surfaces, and solids.

We define a *plane figure* as a figure formed by points and lines all lying in the same plane.

A figure is called *rectilinear* when it is composed of straight lines only.

A straight line is sometimes called a *right line.*

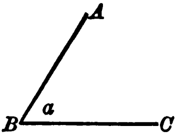
9. *Geometry* treats of the properties, construction, and measurement of geometrical figures.

Plane Geometry treats of plane figures only.

Solid Geometry treats of figures which are not plane.

10. We define an *angle* as the figure formed by two straight lines drawn from a point.

The point is called the *vertex* of the angle, and the straight lines its *sides.*



11. If there is but one angle at a given vertex, we designate it by the letter at the vertex.

But if two or more angles have the same vertex, we avoid ambiguity by naming also a letter on each side, reading the vertex letter between the others.

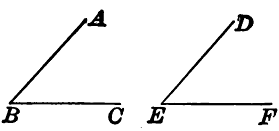
Thus, we should read the angle of § 10 "angle *B*"; but if there were other angles at the same vertex, we should read it either *ABC* or *CBA*.

Another way of designating an angle is by means of a letter placed between its sides; thus, we may read the angle of § 10 "angle *a*."

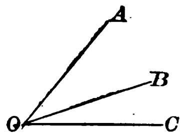
12. Two figures are said to be *equal* when one can be applied to the other so that they shall coincide throughout.

To prove two *angles* equal, we do not consider the lengths of their sides.

Thus, if angle *ABC* can be applied to angle *DEF* in such a manner that point *B* shall fall on point *E*, side *AB* on side *DE*, and side *BC* on side *EF*, the angles are equal, even if side *AB* is not equal in length to side *DE*, and *BC* to *EF*.



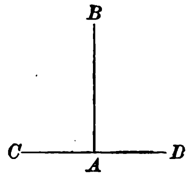
13. We call two angles *adjacent* when they have the same vertex, and a common side between them; as *AOB* and *BOC*.



We call angle *AOC* the *sum* of angles *AOB* and *BOC*.

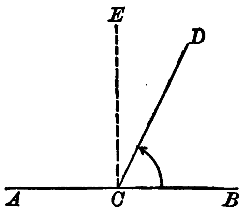
We also regard angle *AOC* as greater than angle *BOC*, and angle *BOC* as less than angle *AOC*.

14. If from a point in a straight line a line be drawn in such a way as to make the adjacent angles equal, each of the adjacent angles is called a *right angle*, and the lines are said to be *perpendicular* to each other.



Thus, if from point *A* in line *CD* line *AB* be drawn in such a way as to make angles *BAC* and *BAD* equal, each of these angles is a right angle, and *AB* and *CD* are perpendicular to each other.

15. Let *C* be any point in straight line *AB*; and let straight line *CD* be drawn in such a way as to make angle *BCD* less than angle *ACD*.



Let line *CD* be turned about point *C* as a pivot towards the position *CA*.

Then, angle *BCD* will constantly increase, and angle *ACD* will constantly diminish; and there must be some position of *CD* where these angles are equal, and there can evidently be but one such position.

If *CE* is this position, by the definition of § 14, *CE* is perpendicular to *AB* at *C*.

Then, *at a given point in a straight line, a perpendicular to the line can be drawn, and but one.*

The following are immediate consequences of § 15:

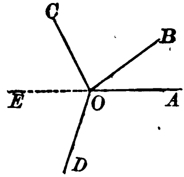
16. *Any two right angles are equal.*

17. *If two adjacent angles have their exterior sides in the same straight line, their sum is equal to two right angles.*

For in figure of § 15, the sum of angles ACD and BCD equals the sum of angles ACE and BCE, or two right angles.

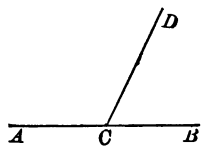
18. *The sum of all the angles on the same side of a straight line at a given point is equal to two right angles.*

19. *The sum of all the angles about a point in a plane is equal to four right angles.*



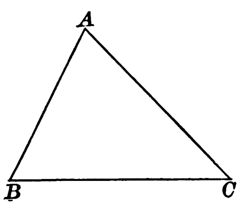
For if a side of any angle, as OA, be extended to E, the sum of the angles on either side of straight line AE is, by § 18, equal to two right angles.

20. *If the sum of two adjacent angles is equal to two right angles, their exterior sides lie in the same straight line.*



If the sum of angles ACD and BCD is two right angles, side AC if extended through C must coincide with CB; for if it did not the sum of the angles would be less or greater than 180° (§ 17).

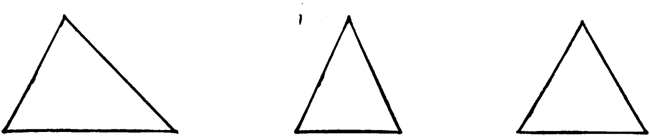
21. We define a *triangle* as a portion of a plane bounded by three straight lines; as ABC.



We call the lines AB, BC, and CA the *sides* of the triangle; and their intersections, A, B, and C, the *vertices*.

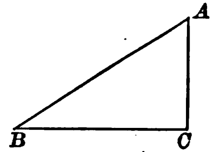
We define the *angles* of the triangle as the angles CAB, ABC, and BCA, between the adjacent sides.

22. A triangle is called *scalene* when no two sides are equal; *isosceles* when two sides are equal; *equilateral* when all its sides are equal; *equiangular* when all its angles are equal.



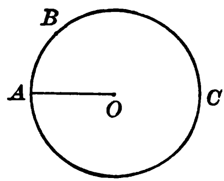
Scalene Isosceles Equilateral

A *right triangle* is a triangle which has a right angle; as *ABC*, which has a right angle at *C*.



The side *AB* opposite the right angle is called the *hypotenuse*, and the other sides, *AC* and *BC*, the *legs*.

23. We define a *circle* as a portion of a plane bounded by a curve, called the *circumference*, all points of which are equally distant from a point within called the *centre*.

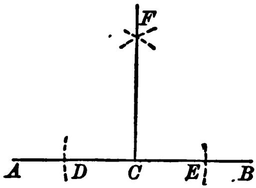


An *arc* is any portion of the circumference; as *AB*.

A *radius* is a straight line drawn from the centre to the circumference; as *OA*.

CONSTRUCTIONS

24. *At a given point in a straight line, to draw a perpendicular to the line.*



Let it be required to draw a line perpendicular to the straight line *AB*, at the point *C*.

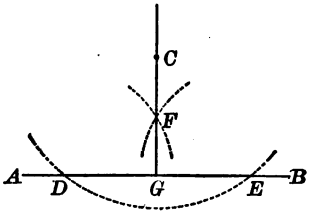
With *C* as centre, and any straight line less than *AC* as radius, describe arcs intersecting *AC* at *D*, and *BC* at *E*.

With points *D* and *E* as centres, and any radius, describe arcs intersecting at *F*.

Then, the straight line drawn from *C* through *F* will be perpendicular to *AB* at *C*.

The reason for the above construction will be found in § 58.

25. *From a given point without a straight line to draw a perpendicular to the line.*



Let it be required to draw through any point C without the straight line AB a line perpendicular to AB.

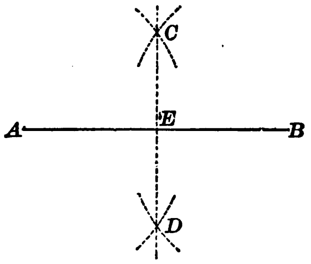
With C as a centre, and any radius, describe an arc cutting AB at points D and E.

With points D and E as centres, and another radius, describe arcs intersecting at F.

Then, the straight line CG drawn from C through F will be perpendicular to AB.

The reason for the construction will be found in § 58.

26. *To bisect a given straight line.*



Let it be required to divide the straight line AB into two equal parts.

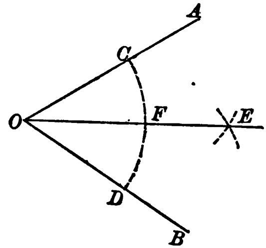
With points A and B as centres, and with the same radius, describe arcs intersecting at C and D.

Draw straight line CD intersecting AB at E.

Then, E is the middle point of AB.

The reason for the construction will be found in § 58.

27. *To bisect a given angle.*



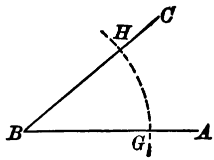
Let it be required to bisect angle AOB.

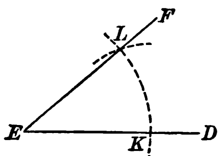
With O as a centre, and any radius, describe an arc intersecting OA at C and OB at D; with C and D as centres, and the same radius, describe arcs intersecting at E.

Then, the straight line OE will bisect angle AOB.

The reason for the construction will be found in § 58.

28. *With a given vertex, and a given side, to construct an angle equal to a given angle.*





Let it be required to construct with E as the vertex, and ED as a side, an angle equal to angle ABC.

With B as a centre, and any radius, describe an arc intersecting AB at G and BC at H.

With E as a centre, and BG as a radius, describe an arc intersecting DE at K.

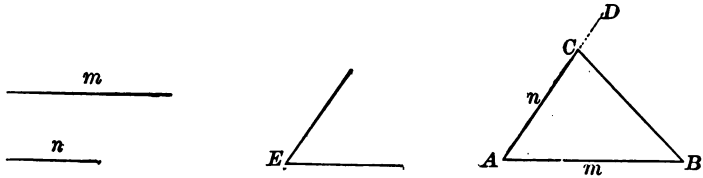
With K as a centre, and the distance from G to H as a radius, describe an arc intersecting the former arc at L.

Draw straight line ELF.

Then, angle DEF will equal angle ABC.

The reason for the construction will be found in § 53.

29. *Given two sides and the included angle of a triangle, to construct the triangle.*



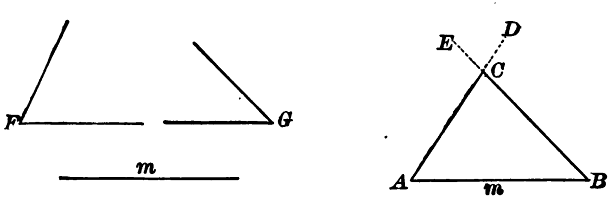
Let it be required to construct the triangle having for two of its sides the straight lines m and n, and their included angle equal to angle E.

Draw line AB equal to m, and construct angle BAD equal to angle E (§ 28).

On AD take AC equal to n, and draw straight line BC.

Then, ABC is the required triangle.

30. *Given a side and two adjacent angles of a triangle, to construct the triangle.*



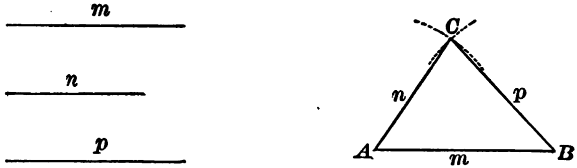
Let it be required to construct the triangle having for a side the straight line m, and its adjacent angles equal to angles F and G.

Draw line AB equal to m, and construct angle BAD equal to angle F (§ 28).

Draw line BE, making angle ABE equal to angle G, intersecting AD at C.

Then, ABC is the required triangle.

31. *Given the three sides of a triangle, to construct the triangle.*



Let it be required to construct the triangle having for its sides the straight lines m, n, and p.

Take the straight line AB equal to m.

With A as a centre, and n as a radius, describe an arc.

With B as a centre, and p as a radius, describe an arc intersecting the former arc at C.

Then, ABC is the required triangle.

32. ~~We define an~~ *Axiom* ~~as a truth which is assumed without proof as being~~ self-evident. Truth

We define a *Theorem* as a truth requiring proof.

We define a *Problem* as a question proposed for solution.

A *Proposition* is a general term for a theorem or a problem.

A *Postulate* assumes that a certain problem can be solved.

A *Corollary* is a truth which is an immediate consequence of the proposition which it follows.

An *Hypothesis* is a supposition, made either in the statement or the proof of a proposition.

33. Postulates.

We assume that the following problems can be solved:

1. A straight line can be drawn between any two points.
2. A straight line can be extended indefinitely in either direction.

34. Axioms.

We assume the following as true:

1. *Things which are equal to the same thing, or to equal things, are equal to each other.*
2. *If equals be added to equals, the sums will be equal.*

3. *If equals be subtracted from equals, the remainders will be equal.*

4. *If equals be multiplied by equals, the products will be equal.*

5. *If equals be divided by equals, the quotients will be equal.*

6. *But one straight line can be drawn between two points.*

7. *A straight line is the shortest line between two points.*

8. *The whole is equal to the sum of all its parts.*

9. *The whole is greater than any of its parts.*

35. Since but one straight line can be drawn between two points, a straight line is said to be *determined* by any two of its points.

36. Symbols and Abbreviations.

The following symbols are used in the work:

$+$, plus.
$-$, minus.
$\times$, multiplied by.
$=$, equals.
$\approx$, equivalent, is equivalent to.
$>$, is greater than.
$<$, is less than.
$\therefore$, therefore.
$\angle$, angle.
$\angle$s, angles.
$\triangle$, triangle.
$\triangle$s, triangles.
$\perp$, perpendicular, is perpendicular to.
$\perp$s, perpendiculars.
$\parallel$, parallel, is parallel to.
$\parallel$s, parallels.
$\square$, parallelogram.
$\square$s, parallelograms.
$\odot$, circle.
$\odot$s, circles.

The following abbreviations are used:

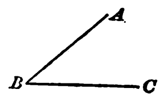
Ax.,	Axiom.
Def.,	Definition.
Hyp.,	Hypothesis.
Cons.,	Construction.
Rt.,	Right.
Str.,	Straight.
Adj.,	Adjacent.
Sup.,	Supplementary.
Alt.,	Alternate.
Int.,	Interior.
Ext.,	Exterior.
Corresp.,	Corresponding.
Rect.,	Rectangle, rectangular.

Book I

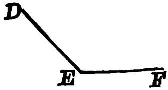
RECTILINEAR FIGURES

DEFINITIONS

37. We define an *acute* angle as an angle less than a right angle; as *ABC*.

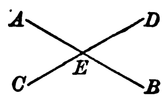


We define an *obtuse* angle as an angle greater than a right angle; as *DEF*.

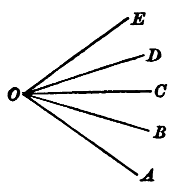


Acute and obtuse angles are called *oblique* angles; and intersecting lines which are not perpendicular, are said to be oblique to each other.

We call two angles *vertical* when the sides of one are the prolongations of the sides of the other; as *AEC* and *BED*.



38. If angles *AOB, BOC, COD, DOE,* are all equal, we say that angle *AOB* is contained *four times* in angle *AOE*; and similarly for any number of equal parts of angle *AOE*.



39. We *measure* an angle by finding how many times it contains another angle taken as the unit of measure.

The usual unit of measure for angles is the *degree,* which is the ninetieth part of a right angle.

To express fractional parts of the unit, we divide the degree into sixty equal parts, called *minutes,* and the minute into sixty equal parts, called *seconds.*

We represent degrees, minutes, and seconds by the symbols °, ′, ″, respectively.

40. If the sum of two angles is a right angle, or 90°, we call one the *complement* of the other; if their sum is two right angles, or 180°, we call one the *supplement* of the other.

Thus, the complement of an angle of 34° is 90° – 34°, or 56°; the supplement of an angle of 34° is 180° – 34°, or 146°.

Two angles which are complements of each other are called *complementary;* and two angles which are supplements of each other are called *supplementary.*

41. It follows from the above that:

1. *The complements of equal angles are equal.*
2. *The supplements of equal angles are equal.*

42. Since ∠s *ACD, BCD* (Fig., § 15), are *supplementary* (§ 40), the principle of § 17 may be stated as follows:

If two adjacent angles have their exterior sides in the same straight line, they are supplementary.

Such angles are called *supplementary-adjacent.*

Ex. 1. How many degrees are there in the complement of 43°? of 85°?

Ex. 2. How many degrees are there in the supplement of 17°? of 162°?

Ex. 3. Find the supplement of the complement of 75°; of 22° 30′; of 45° 12′ 18″; of A°.

Ex. 4. Find the complement of the supplement of 178°; of 144°; of 125° 14′ 15″; of B°.

Ex. 5. Given the sum and difference of two lines, to find the lines. (Bisect the sum, also bisect the difference.)

Ex. 6. The sum of the lengths of two given lines is 20 inches, and their difference is 4 inches; find the lines. Make a diagram.

43. Note. The demonstration of a geometric truth consists of three parts:

1. *The statement of what is given in the figure.*
2. *The statement of what is to be proved.*
3. *The proof, depending upon previous theorems, axioms, or definitions.*

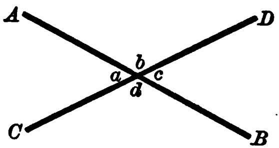
We shall mark these three divisions of the demonstration by the words **Given, To Prove**, and **Proof**, respectively.

We shall use the symbols and abbreviations of § 36, and also number the successive steps of the proof.

In every proof in Book I, where a preceding theorem is given as the authority for any statement in the proof, it will be found directly after the statement, in smaller type, enclosed in brackets.

PROP. I. THEOREM

44. *If two straight lines intersect, the vertical angles are equal.*



Draw intersecting str. lines AB and CD forming vertical $\angle s$ a and c, and also b and d. We then have:

Given intersecting str. lines AB and CD, forming vertical $\angle s$ a and c, and also b and d.

To Prove $\angle a = \angle c$ and $\angle b = \angle d$.

Proof. 1. $\angle a + \angle b = 180°$.

[If two adj. $\angle s$ have their ext. sides in the same str. line, their sum is equal to two rt. $\angle s$.] (§ 17)

2. Also, $\angle b + \angle c = 180°$.

3. Then, $\angle a + \angle b = \angle b + \angle c$.

[Things which are equal to the same thing, or to equal things, are equal to each other.] (Ax. 1)

4. Subtracting $\angle b$ from the equals $\angle a + \angle b$ and $\angle b + \angle c$,

$$\angle a = \angle c.$$

[If equals be subtracted from equals, the remainders will be equal.] (Ax. 3)

5. In like manner, $\angle b = \angle d$.

45. The enunciation of every theorem consists essentially of two parts: the *Hypothesis*, and the *Conclusion*.

Thus, we may enunciate Prop. I as follows:

Hypothesis. If two straight lines intersect,

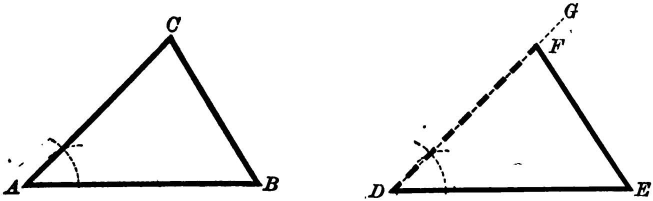
Conclusion. The vertical angles are equal.

PROP. II.

PLATE I

PROP. II. THEOREM

46. *Two triangles are equal when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*



Draw any $\triangle ABC$: draw line DE equal to AB: construct $\angle EDG$ equal to $\angle A$, the constructed side being DG: on DG take DF equal to AC: draw line FE. We now have:

Given, in $\triangle$s ABC and DEF,

$$AB = DE,\ AC = DF, \text{ and } \angle A = \angle D.$$

To Prove $\triangle ABC = \triangle DEF$.

Proof. 1. Superpose $\triangle ABC$ upon $\triangle DEF$ in such a way that $\angle A$ shall coincide with its equal $\angle D$; side AB falling on side DE, and side AC on side DF.

2. Since $AB = DE$, point B will fall on point E.

3. Since $AC = DF$, point C will fall on point F.

4. Then, side BC will fall on side EF.

[But one str. line can be drawn between two points.] (Ax. 6)

5. Therefore, the $\triangle$s coincide throughout, and are equal.

47. Since ABC and DEF coincide throughout, we have

$$\angle B = \angle E,\ \angle C = \angle DFE, \text{ and } BC = EF.$$

48. Note. In equal figures, lines or angles which are similarly placed are called *homologous*.

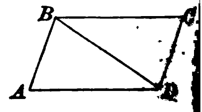
Thus, in the figure of Prop. II, $\angle A$ is homologous to $\angle D$; AB is homologous to DE; etc.

It follows from the above that

In equal figures, the homologous parts are equal.

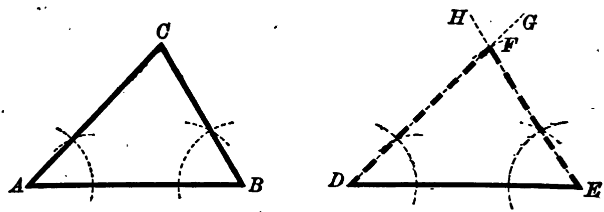
In equal triangles, the equal sides lie opposite the equal angles.

Ex. 7. If, in triangles ABD and BCD, sides AB and BD and angle ABD are equal respectively to sides CD, BD, and angle BDC, what other sides and angles of the triangles are equal?



Prop. III. Theorem

49. *Two triangles are equal when a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other.*



Draw any $\triangle ABC$: draw line DE equal to AB: at D construct $\angle EDG$ equal to $\angle A$: at E construct $\angle DEH$ equal to $\angle B$. Let F be point of intersection of DG and EH. We now have:

Given, in $\triangle$s ABC and DEF,

$$AB = DE,\ \angle A = \angle D, \text{ and } \angle B = \angle E.$$

To Prove $\triangle ABC = \triangle DEF.$

Proof. 1. Superpose $\triangle ABC$ upon $\triangle DEF$ in such a way that side AB shall coincide with its equal DE; point A falling on point D, and point B on point E.

2. Since $\angle A = \angle D$, side AC will fall on side DF, and point C will fall somewhere on DF.

3. Since $\angle B = \angle E$, side BC will fall on side EF, and point C will fall somewhere on EF.

4. Since point C falls at the same time on DF and EF, it must fall at their intersection, F.

5. Then, the $\triangle$s coincide throughout and are equal.

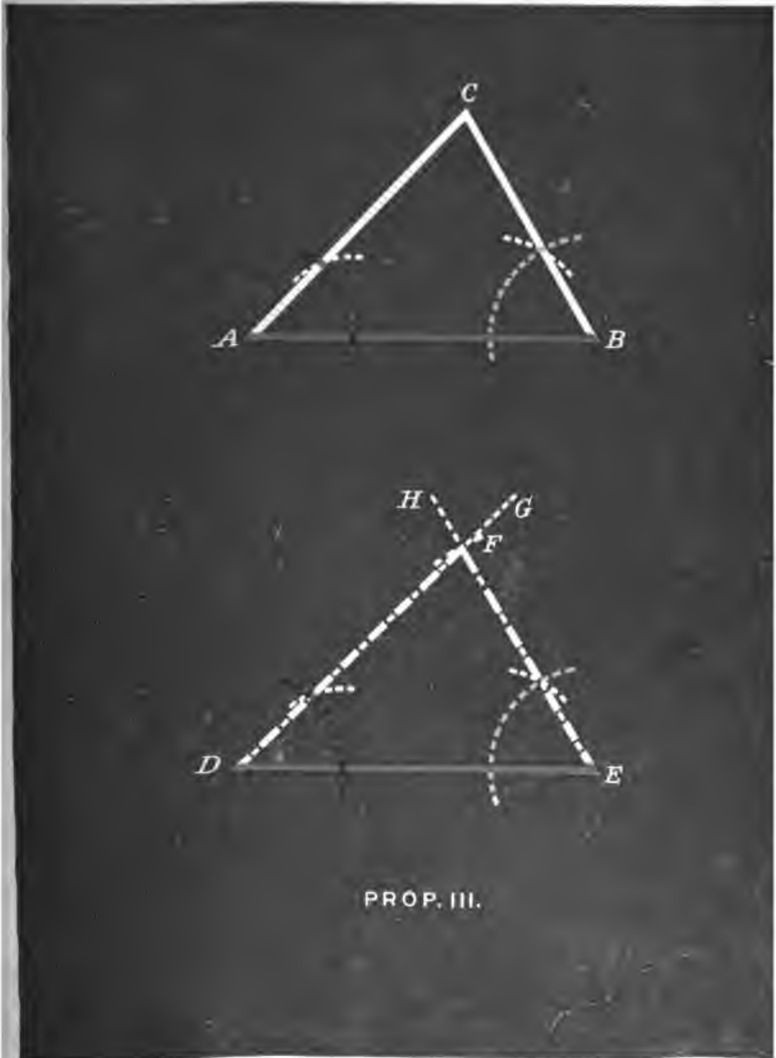
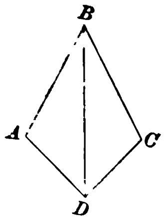


PLATE II

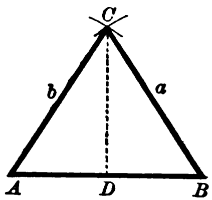


Ex. 8. What other sides and angles in the triangles of Prop. III are equal?

Ex. 9. If, in triangles ABD and BCD, side BD and angles ABD and ADB are equal respectively to side BD and angles CBD and CDB, what other sides and angles of the triangles are equal?

PROP. IV. THEOREM

50. *In an isosceles triangle, the angles opposite the equal sides are equal.*



Draw line AB. With any radius greater than $\frac{1}{2}AB$ and with A as a centre draw an arc; with the same radius and B as a centre draw an arc intersecting the first arc at C; draw lines $AC\,(b)$ and $BC\,(a)$. We then have:

Given a and b, the equal sides of isosceles $\triangle ABC$.

To Prove $\angle A = \angle B$.

Proof. 1. Draw str. line CD bisecting $\angle ACB$, meeting AB at D.

2. In $\triangle$s ACD and BCD, $CD = CD$.

3. By hyp., $a = b$, and $\angle ACD = \angle BCD$.

4. Then, $\triangle ACD = \triangle BCD$.

[Two $\triangle$s are equal when the two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 46)

5. Then, $\angle A = \angle B$.

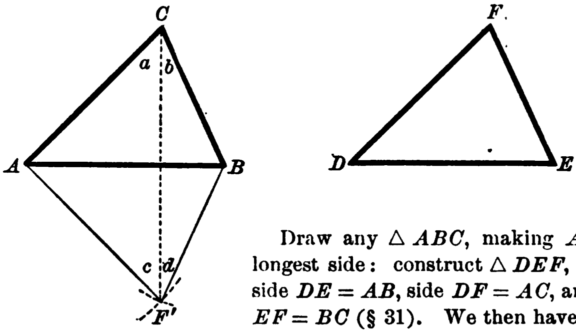
[In equal figures, the homologous parts are equal.] (§ 48)

51. It follows from the preceding that

An equilateral triangle is also equiangular.

Prop. V. Theorem

52. *Two triangles are equal when the three sides of one are equal respectively to the three sides of the other.*



Draw any $\triangle ABC$, making AB the longest side: construct $\triangle DEF$, having side $DE = AB$, side $DF = AC$, and side $EF = BC$ (§ 31). We then have:

Given in $\triangle$s ABC and DEF, $AB = DE$, $BC = EF$, and $AC = DF$.

To Prove $\triangle ABC = \triangle DEF$.

Proof. 1. Let AB be longest side of $\triangle ABC$; and place $\triangle DEF$ in position ABF', side DE coinciding with its equal AB, and vertex F falling at F', on the opposite side of AB from C.

2. Draw line CF', and represent $\angle$s ACF', BCF', $AF'C$, and $BF'C$ by a, b, c, and d, respectively.

3. Since $AC = AF'$, $\angle a = \angle c$. (1)

[In an isosceles $\triangle$, the $\angle$s opposite the equal sides are equal.] (§ 51)

4. Since $BC = BF'$, $\angle b = \angle d$. (2)

5. Adding (1) and (2),

$$\angle a + \angle b = \angle c + \angle d; \text{ or } \angle ACB = \angle AF'B.$$

6. Since sides AC and BC and $\angle ACB$, of $\triangle ABC$, are equal, respectively, to sides AF' and BF' and $\angle AF'B$, of $\triangle ABF'$,

$$\triangle ABC = \triangle ABF'.$$

[Two $\triangle$s are equal when two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 46)

7. That is, $\triangle ABC = \triangle DEF$.

53. Note. We may now see the reason for the construction given in § 28. We know that the triangle whose vertices are the points B, G, and H has its sides respectively equal to those of the triangle whose vertices are the points E, K, and L.

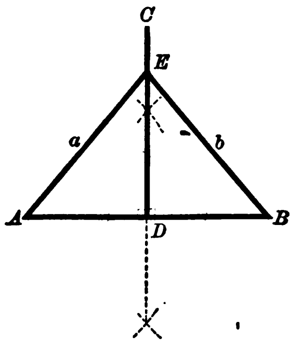
Then, the triangles are equal by § 52, and the homologous angles B and E are equal.

Prop. VI. Theorem

54. *If a perpendicular be erected at the middle point of a straight line,*

I. *Any point in the perpendicular is equally distant from the extremities of the line.*

II. *Any point without the perpendicular is unequally distant from the extremities of the line.*



At the middle point D of any line AB erect a $\perp$ DC. From E, any point in DC, draw lines to A and B. We now have:

I. **Given** line $CD \perp$ to line AB at its middle point D, E any point in CD, and lines AE (a) and BE (b).

To Prove $a = b$.

Proof. 1. In $\triangle$ ADE and BDE, $DE = DE$.

2. By hyp., $AD = DB$.

3. Also, $\angle ADE = \angle BDE$.

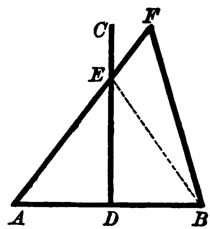
[All rt. $\angle$s are equal.] (§ 16)

4. Then, $\triangle ADE = \triangle BDE$.

[Two $\triangle$ are equal when two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 46)

5. Then, $a = b$.

[In equal figures, the homologous parts are equal.] (§ 48)



Draw line AB. At D, the middle point of AB, draw $DC \perp AB$. From F, any point without CD, draw lines FA and FB. We then have:

II. **Given** line $CD \perp$ to line AB at its middle point D, F any point without CD, and lines AF and BF.

To Prove AF and BF unequal.

Proof. 1. Let AF intersect CD at E, and draw line BE.

2. We have $BE + EF > BF$.

[A str. line is the shortest line between two points.] (Ax. 7)

3. But, $BE = AE$.

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 54, I)

4. Substituting for BE its equal AE,

$$AE + EF > BF, \text{ or } AF > BF.$$

55. It follows from Prop. VI that *every point which is equally distant from the extremities of a straight line, lies in the perpendicular erected at the middle point of the line.*

56. A straight line is determined by any two of its points (§ 35); whence,

Two points, each equally distant from the extremities of a straight line, determine a perpendicular to that line at its middle point.

57. From the equal triangles ADE and BDE, in the figure of § 54, I,

$$\angle AED = \angle BED;$$

since homologous parts of equal figures are equal.

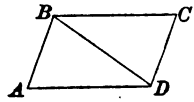
Then, *if lines be drawn to the extremities of a straight line from any point in the perpendicular erected at its middle point, they make equal angles with the perpendicular.*

58. We may now see the reasons for the constructions given in §§ 24, 25, 26, and 27.

In § 24, by construction, points C and F are each equally distant from points D and E; and CF is perpendicular to AB by § 56.

In § 25, points C and F are each equally distant from points D and E; and in § 26, points C and D are each equally distant from points A and B.

In § 27, points O and E are each equally distant from points C and D; then, OE is perpendicular to straight line CD at its middle point (§ 56), and $\angle AOE = \angle BOE$ by § 57.



Ex. 10. If in triangles ABD and BCD, sides AB, BD, and AD are equal, respectively, to sides CD, BD, and BC, what angles of the triangles are equal?

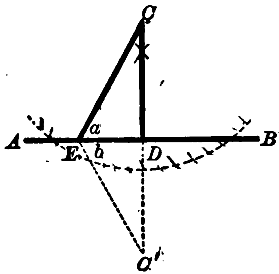
Ex. 11. Two lines of unequal length bisect each other at right angles. Show that any point in either line is equidistant from the extremities of the other. (§ 54.)

Ex. 12. If lines be drawn from the extremities of a straight line to any point in the perpendicular erected at its middle point, they make equal angles with the line. (Figure of § 54, I. Use § 48.)

Prop. VII. Theorem

59. *From a given point without a straight line, but one perpendicular can be drawn to the line.*

(It follows from § 25 that, from a given point without a straight line, a perpendicular can be drawn to the line.)



From point C without line AB, draw $CD \perp AB$ (§ 25). We then have:

~~Given~~ point C without line AB, and line $CD \perp AB$.

~~To Prove~~ CD the only $\perp$ which can be drawn from C to AB

Proof. 1. If possible, let CE be another $\perp$ from C to AB.

2. Extend CD to C', making $C'D = CD$, and draw line EC'; represent $\angle CED$ by a, and $\angle C'ED$ by b.

3. Since ED is $\perp CC'$ at its middle point D, $\angle a = \angle b$.

[If ~~lines be drawn~~ to ~~the~~ extremities of a str. line from any point ~~in~~ the $\perp$ erected at its middle point, they make equal $\angle$s with the $\perp$.] (§ 57)

4. By hyp., $\angle a$ is a rt. $\angle$.

5. Then, $\angle b$ is a rt. $\angle$, and $\angle a + \angle b =$ two rt. $\angle$s.

6. Then line CEC' is a str. line.

[If ~~the sum of two~~ adj. $\angle$s is equal to two rt. $\angle$s, their ext. sides lie in ~~the same~~ str. line.] (§ 20)

7. This is impossible, for, by cons., CDC' is a str. line.

[But one str. line can be drawn between two points.] (Ax 6.)

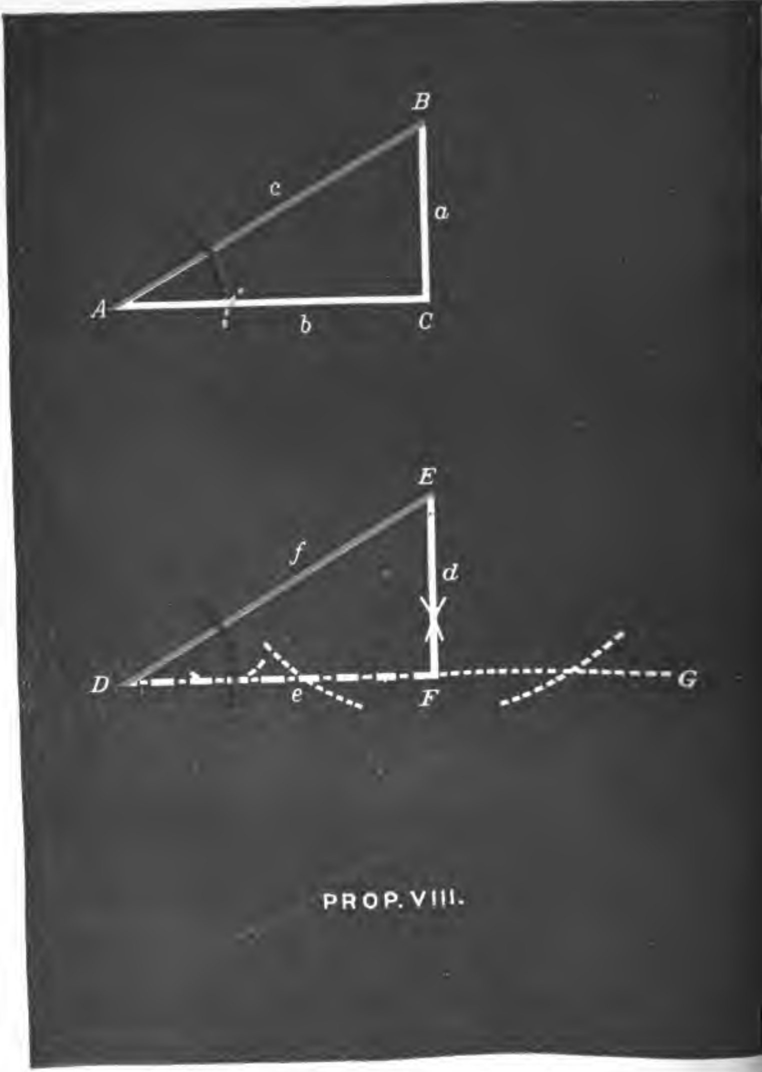


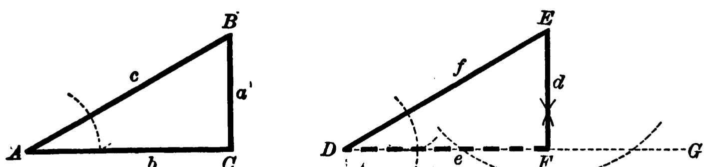
PLATE III

8. Hence, CE cannot be $\perp$ AB, and CD is the only $\perp$ that can be drawn.

Ex. 13. If AB, AC, and DB, DC are the equal sides of two unequal isosceles triangles ABC, DBC, having common side BC, the line joining A and D bisects BC.

Prop. VIII. Theorem

60. *Two right triangles are ~~equal~~ when the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other.*



Draw any $\triangle$ ABC, right angled at C; draw line $DE = AB$; construct $\angle D = \angle A$; draw line $EF \perp DG$. We now have:

Hyp ~~**Given**~~ rt. $\triangle$ ABC and DEF,

hypotenuse AB (c) = hypotenuse DE (f),

and $\angle A = \angle D$.

Con ~~**To Prove**~~ $\triangle ABC \cong \triangle DEF$.

Proof. 1. Denote BC ~~by a~~, AC ~~by b~~, EF ~~by d~~, and DF ~~by e~~.

2. Superpose $\triangle$ ABC upon $\triangle$ DEF in such a way that hypotenuse c shall coincide with its equal f; vertex A falling on vertex D, and vertex B on vertex E.

3. Since $\angle A = \angle D$, side b will fall on side e.

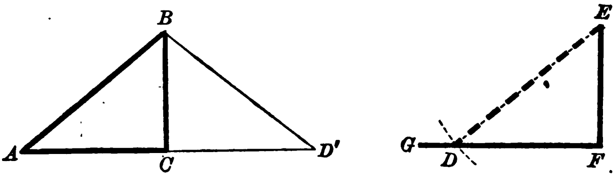
4. Then, side a will fall on side d.

[From a given point without a str. line, but one $\perp$ can be drawn to the line.] (§ 59)

5. Therefore, the $\triangle$ coincide throughout, and are equal.

Prop. IX. Theorem

61. *Two right triangles are ~~equal~~ when the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.*



Draw any $\triangle ABC$, right angled at C; draw line $EF = BC$; construct rt. $\angle EFG$; with E as centre and AB as radius describe an arc intersecting FG at D; draw line DE. We now have:

~~Given~~, in rt. $\triangle$s ABC and DEF,

hypotenuse AB = hypotenuse DE, and $BC = EF$.

~~To Prove~~ $\triangle ABC = \triangle DEF$.

Proof. 1. Place $\triangle DEF$ in position BCD'; vertex E falling at B, F at C, and D at D', on opposite side of BC from A.

2. Since ACB and BCD' are rt. $\angle$s, ACD' is a str. ~~line~~.

[If the sum ~~of two~~ adj. $\angle$s is equal to two ~~rt. $\angle$s, their ext. sides lie~~ in the same ~~str. line~~.] (§ 20)

3. Then, $\triangle ABD'$ is isosceles, and $\angle A = \angle D'$.

[In an ~~isosceles~~ $\triangle$, ~~the $\angle$s opposite~~ the equal sides are equal.] (§ 50)

4. Then, $\triangle ABC = \triangle BCD'$.

[Two rt. $\triangle$s are equal when the hypotenuse and an adj. $\angle$ of one are equal respectively to the hypotenuse and an adj. $\angle$ of the other.] (§ 60)

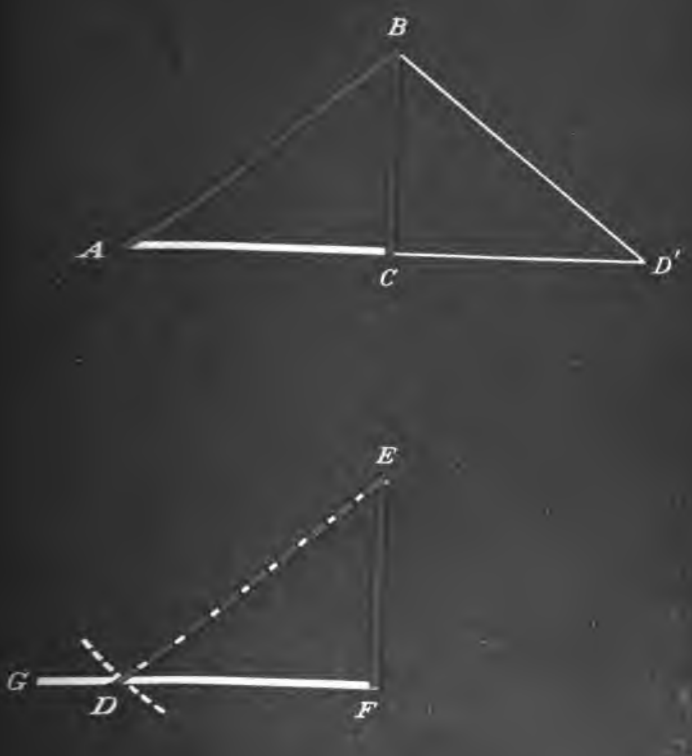
5. That is, $\triangle ABC = \triangle DEF$.

Prop. X. Theorem

62. *Any side of a triangle is greater than the difference of the other two sides.*

Draw $\triangle ABC$ with side $BC >$ side AC. We now have:

Given AB, any side of $\triangle ABC$, and $BC > AC$.



PROP. IX.

PLATE IV

To Prove $AB > BC - AC.$

Proof. 1. We have $AB + AC > BC.$

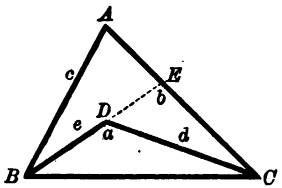
[A str. line is the shortest line between two points.] (Ax. 7)

2. Subtracting AC from both members of the inequality,

$$AB > BC - AC.$$

PROP. XI. THEOREM

63. *The sum of any two sides of a triangle is greater than the ~~sum of~~ the lines drawn from any point within the triangle to the extremities of the remaining side.*



Draw $\triangle ABC$. From D any point within the $\triangle$ draw lines DC and DB. Represent AC, AB, DC, and DB by b, c, d, and e, respectively. We then have:

Given d and e lines drawn from any point D within $\triangle ABC$ to the extremities of side BC.

To Prove $b + c > d + e.$

Proof. 1. Extend BD to meet AC at E.

2. We have $c + AE > e + DE.$ (1)

[A str. line is the shortest line between two points.] (Ax. 7)

3. For same reason, $CE + DE > d.$ (2)

4. Adding inequalities (1) and (2), and observing that $AE + CE = b$, we have

$$b + c + DE > d + e + DE.$$

5. Subtracting DE from both members of the inequality,

$$b + c > d + e.$$

Ex. 14. Given one side of a triangle, and the perpendicular drawn to it from the vertex of the opposite angle, to construct the triangle. Can more than one such triangle be drawn?

Ex. 15. There are six elements in every triangle, — three sides and three angles. How many of these elements and what elements are needed that only one definite triangle can be constructed?

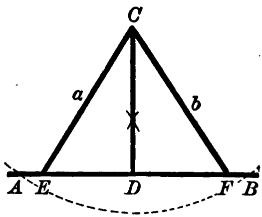
Ex. 16. Triangles are to be formed by choosing any three lines from six given lines whose lengths are 2, 3, 4, 5, 6, 7 inches, respectively. How many triangles can you form and what kind of triangles are they?

Prop. XII. Theorem

64. *If oblique lines be drawn from a point to a straight line,*

I. *Two oblique lines cutting off equal distances from the foot of the perpendicular from the point to the line are equal.*

II. *Of two oblique lines cutting off unequal distances from the foot of the perpendicular from the point to the line, the more remote is the greater.*



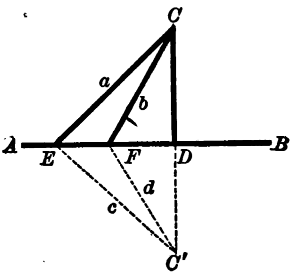
I. ~~From~~ any point C ~~without~~ line AB, draw $CD \perp AB$ (§ 25); ~~take~~ line $ED = DF$ and draw lines CE (a) and CF (b). We then have:

I. **~~Given~~** $CD \perp$ from point C to line AB; and lines a and b cutting off equal distances from the foot of CD.

~~To Prove~~ $a = b$.

Proof. Since CD is $\perp EF$ at its middle point D, $a = b$.

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the ~~extremities~~ of the line.] (§ 54)



II. Construct the figure in accordance with the statement. We then have:

II. **Given** CD the $\perp$ from point C to line AB; and CE (a) and CF (b) oblique lines from C to AB, cutting off unequal distances from the foot of CD; a being the more remote.

To Prove $a > b$.

Proof. 1. Produce CD to C', making $C'D = CD$, and draw lines $C'E$ (c) and $C'F$ (d).

2. Since by cons., AD is $\perp$ CC' at its middle point D,

$$a = c, \text{ and } b = d.$$

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 54)

3. But $a + c > b + d$.

[The sum of any two sides of a $\triangle$ is > the sum of the lines drawn from any point within the $\triangle$ to the extremities of the remaining side.] (§ 63)

4. Substituting for c and d their equals a and b, respectively,

$$2a > 2b.$$

5. Dividing by 2, $a > b$.

Note. The theorem holds equally if CE is on the opposite side of CD from CF.

Ex. 17. Two sides of a triangle are 8 inches and 10 inches, respectively. Between what limits must the third side lie?

Prop. XIII. Theorem

65. *If oblique lines be drawn from a point to a straight line,*

I. *Two equal oblique lines cut off equal distances from the foot of the perpendicular from the point to the line.*

II. *Of two unequal oblique lines, the greater cuts off the greater distance from the foot of the perpendicular from the point to the line.*

I. Draw line AB and take C any point without AB. Draw line $CD \perp AB$. With C as centre and radius greater than CD describe arc intersecting AD at E, and BD at F; draw lines CE (a) and CF (b). We then have:

I. **Given** $CD \perp$ from point C to line AB, and a and b equal oblique lines from C to AB.

To Prove $ED = DF$.

Proof. 1. In rt. $\triangle$ CDE and CDF, $CD = CD$.

2. By hyp., $a = b$.

3. Then, $\triangle CDE = \triangle CDF$.

[Two rt. $\triangle$ are equal when the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.] (§ 61)

4. Then, $DE = DF$.

[In equal figures, the homologous parts are equal.] (§ 48)

II. Draw line AB and take C any point without AB. Draw line $CD \perp AB$; draw lines CE (a) and CF (b) intersecting AD at E and F, respectively, making $a > b$. We then have:

II. **Given** CD the $\perp$ from point C to line AB; and a and b unequal oblique lines from C to AB, a being $> b$.

To Prove $DE > DF$.

Proof. 1. We know that DE is either $<, =,$ or $> DF$.

2. If we suppose $DE < DF$, a would be $< b$.

[If oblique lines be drawn from a point to a str. line, of two oblique lines cutting off unequal distances from the foot of the $\perp$ from the point to the line, the more remote is the greater.] (§ 64)

3. But this is contrary to the hypothesis that a is $> b$; then DE cannot be $< DF$.

4. If we suppose $DE = DF$, a would be $= b$.

[If oblique lines be drawn from a point to a str. line, two oblique lines cutting off equal distances from the foot of the $\perp$ from the point to the line are equal.] (§ 64)

5. This is contrary to the hypothesis that a is $> b$; then DE cannot be $= DF$.

6. If DE cannot be $< DF$, nor $= DF$, we must have

$$DE > DF.$$

66. Note I. The method of proof used in Prop. XIII is called the Indirect Method, or the *Reductio ad absurdum.*

We prove a proposition by making every possible supposition in regard to it; and showing that, in every case except the one we wish to prove, the supposition leads to something contrary to the hypothesis.

67. Note II. We may state Prop. XII, I, as follows:

Hypothesis. If two oblique lines be drawn from a point to a straight line, cutting off equal distances from the foot of the perpendicular from the point to the line,

Conclusion. They are equal.

Again, we may state Prop. XIII, I,

Hypothesis. If two equal oblique lines be drawn from a point to a straight line,

Conclusion. They cut off equal distances from the foot of the perpendicular from the point to the line.

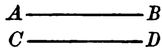
We call one proposition the **converse of** another when the hypothesis and conclusion of the first are, respectively, the conclusion and hypothesis of the second.

It follows from the above that Prop. XIII, I, is the *converse* of Prop. XII, I.

Prop. XIII, II, is the converse of Prop. XII, II; also § 20 is the converse of § 17.

PARALLEL LINES

68. Def. Two straight lines are said to be *parallel* (∥) when they lie in the same plane, and cannot meet however far they may be produced; as AB and CD.



69. Ax. *We assume that but one straight line can be drawn through a given point parallel to a given straight line.*

Prop. XIV. Theorem

70. *Two perpendiculars to the same straight line are parallel.*

Draw a line EG; at points A and C, respectively, on EG, draw lines AB and $CD \perp EG$. We now have:

Given lines AB and $CD \perp$ to line AC.

To Prove $AB \parallel CD$.

Proof. 1. If AB and CD are not ∥, they will meet in some point if sufficiently produced (§ 68).

2. We should then have two ⊥s from this point to AC, which is impossible.

[From a given point without a str. line, but one ⊥ can be drawn to the line.] (§ 59)

3. Therefore, AB and CD cannot meet, and are ∥.

Prop. XV. Theorem

71. *Two straight lines parallel to the same straight line are parallel to each other.*

Draw line EF. Draw lines AB and CD ∥ to EF (§ 70). We then have:

Given lines AB and CD ∥ to line EF.

To Prove $AB \parallel CD$.

Proof. 1. If AB and CD are not ∥, they will meet in some point if sufficiently produced. (§ 68)

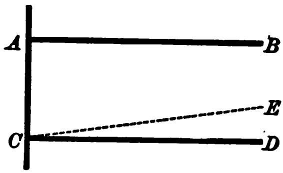
2. We should then have two lines drawn through this point ∥ to EF, which is impossible.

[But one str. line can be drawn through a given point ∥ to a given str. line.] (§ 69)

3. Therefore, AB and CD cannot meet, and are ∥.

Prop. XVI. Theorem

72. *A straight line perpendicular to one of two parallels is perpendicular to the other.*



Draw line $AB \parallel CD$ (§ 70); draw line $AC \perp AB$. We then have:

Given lines AB and CD $\parallel$, and line $AC \perp AB$.

To Prove $AC \perp CD$.

Proof. 1. If CD is not $\perp AC$, let line CE be $\perp AC$.

2. Then since AB and CE are $\perp AC$, $CE \parallel AB$.

[Two ⊥s to the same str. line are ∥.] (§ 70)

3. But by hyp., $CD \parallel AB$.

4. Then, CE must coincide with CD.

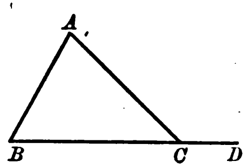
[But one str. line can be drawn through a given point ∥ to a given str. line.] (§ 69)

5. Now by cons., $AC \perp CE$.

6. Then since CE coincides with CD, we have $AC \perp CD$.

DEFINITIONS

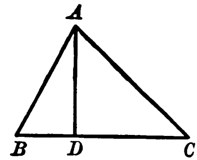
73. An *exterior angle* of a triangle is the angle at any vertex formed by any side of the triangle and the adjacent side produced; as ACD.



If any side of a triangle be taken and called the *base*, we define the corresponding *altitude* as the perpendicular drawn from the opposite vertex to the base, produced if necessary.

In general, any side may be taken as the base; but in an isosceles triangle, unless otherwise specified, the side which is not one of the equal sides is taken as the base.

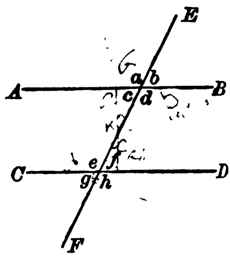
When any side has been taken as the base, we call the



opposite angle the *vertical angle,* and its vertex the *vertex of the triangle.*

Thus, in triangle ABC, BC is the base, AD the altitude, and BAC the vertical angle.

74. If two straight lines, AB and CD, are cut by a line EF, called a *transversal,* the angles are named as follows:



c, d, e, and f are called *interior* angles, and a, b, g, and h *exterior* angles.

c and f, or d and e, are called *alternate-interior* angles.

a and h, or b and g, are called *alternate-exterior* angles.

a and e, b and f, c and g, or d and h, are called *corresponding* angles.

Prop. XVII. Theorem

75. *If two parallels are cut by a transversal, the alternate-interior angles are equal.*

Draw line $AB \parallel CD$ (§ 70). Draw any line EF (not $\perp AB$) intersecting AB and CD at G and H, respectively. We then have:

Given $\parallel$s AB and CD cut by transversal EF at points G and H, respectively, forming alt. int. $\angle$s AGH (a) and GHD (c), also BGH (b) and CHG (d).

To Prove $\angle a = \angle c$, and $\angle b = \angle d$.

Proof. 1. Through K, the middle point of GH, draw a line $\perp AB$, meeting AB at L, and CD at M.

2. Then, $LM \perp CD$.

[A str. line $\perp$ to one of two $\parallel$s is $\perp$ to the other.] (§ 72)

3. In rt. $\triangle$s GKL and HKM, by cons.,

$$\text{hypotenuse } GK = \text{hypotenuse } HK.$$

4. Also, $\angle GKL = \angle HKM$.

[If two str. lines intersect, the vertical $\angle$s are equal.] (§ 44)

5. Then $\triangle GKL = \triangle HKM.$

[Two rt. $\triangle$s are equal when the hypotenuse and an adj. $\angle$ of one are equal respectively to the hypotenuse and an adj. $\angle$ of the other.] (§ 60)

6. Then, $\angle a = \angle c.$

[In equal figures, the homologous parts are equal.] (§ 48)

7. Again, $\angle a$ is the supplement of $\angle b$, and $\angle c$ of $\angle d$.

[If two adj. $\angle$s have their ext. sides in the same str. line, their sum is equal to two rt. $\angle$s.] (§ 17)

8. Then, since $\angle a = \angle c$, we have $\angle b = \angle d$.

[The supplements of equal $\angle$s are equal.] (§ 41)

76. We have $\angle a = \angle BGE$. (Fig. of Prop. XVII.)

[If two str. lines intersect, the vertical $\angle$s are equal.] (§ 44)

Then, since $\angle a = \angle c$, we have $\angle BGE = \angle c$.

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

In like manner,

$\angle AGE = \angle d$, $\angle CHF = \angle a$, and $\angle DHF = \angle b$.

That is, *if two parallels are cut by a transversal, the corresponding angles are equal.*

77. We have $\angle a + \angle b =$ two rt. $\angle$s. (Fig. of Prop. XVII.)

[If two adj. $\angle$s have their ext. sides in the same str. line, their sum is equal to two rt. $\angle$s.] (§ 17)

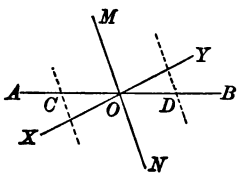
Putting for $\angle b$ its equal $\angle d$, we have

$$\angle a + \angle d = \text{two rt. } \angle\text{s}.$$

In like manner, $\angle b + \angle c =$ two rt. $\angle$s.

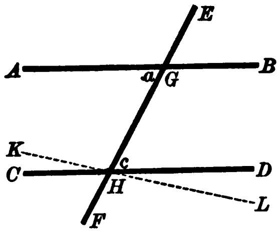
That is, *if two parallels are cut by a transversal, the sum of the interior angles on the same side of the transversal is equal to two right angles.*

Ex. 18. A line AB intersects line XY at O, and through O any line MN is drawn; if through points C and D on AB equidistant from O, parallels to MN be drawn, the triangles formed are equal.



Prop. XVIII. Theorem

78. (Converse of Prop. XVII.) *If two straight lines are cut by a transversal, and the alternate-interior angles are equal, the two lines are parallel.*



Draw line AB. Draw line EF cutting AB at G. Through any point H, in EF, draw line CD making $\angle GHD$ (c) equal $\angle AGH$ (a). We then have:

Given lines AB and CD cut by transversal EF at points G and H, respectively, and

$$\angle a = \angle c.$$

To Prove $AB \parallel CD$.

Proof. 1. If CD is not $\parallel AB$, draw line KL through $H \parallel AB$.

2. Then since $\parallel$s AB and KL are cut by transversal EF,

$$\angle a = \angle GHL.$$

[If two $\parallel$s are cut by a transversal, the alt. int. $\angle$s are equal.] (§ 75)

3. But by hyp., $\angle a = \angle c.$

4. Then, $\angle GHL = \angle c.$

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

5. But this is impossible unless KL coincides with CD.

6. Then, $CD \parallel AB$.

In like manner it may be proved that if AB and CD are cut by EF, $BGH = \angle CHG$, then $AB \parallel CD$.

79. (Converse of § 76.) *If two straight lines are cut by a transversal, and the corresponding angles are equal, the two lines are parallel.*

Suppose $\angle a = \angle CHF.$ (Fig. of Prop. XVIII.)

Now, $\angle c = \angle CHF.$

[If two str. lines intersect, the vertical $\angle$s are equal.] (§ 44)

Then, $\angle a = \angle c.$

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

Then, by § 78, $AB \parallel CD.$

In like manner we may prove that if

$$\angle BGE = \angle c, \text{ or } \angle AGE = \angle CHG, \text{ or } \angle BGH = \angle DHF,$$

then, $AB \parallel CD.$

80. (Converse of § 77.) *If two straight lines are cut by a transversal, and the sum of the interior angles on the same side of the transversal is equal to two right angles, the two lines are parallel.*

Suppose

$$\angle BGH + \angle c = \text{two rt. } \angle\text{s}. \quad \text{(Fig. of Prop. XVIII.)}$$

Then, both $\angle a$ and $\angle c$ are supplements of $\angle BGH$,

and $\angle a = \angle c.$

[The supplements of equal $\angle$s are equal.] (§ 41)

Then, by § 78, $AB \parallel CD.$

In like manner it may be proved that if

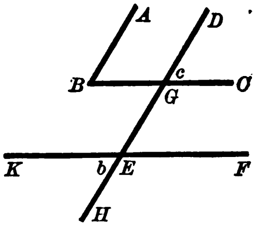
$$\angle a + \angle CHG = \text{two rt. } \angle\text{s, then, } AB \parallel CD.$$

Ex. 19. If two parallels are cut by a transversal, the alternate exterior angles are equal.

Ex. 20. If two straight lines are cut by a transversal, and the alternate-exterior angles are equal, the two lines are parallel.

Prop. XIX. Theorem

81. *Two angles whose sides are parallel, each to each, are equal if both pairs of parallel sides extend in the same direction, or in opposite directions, from their vertices.*



Draw lines AB and BC ∥ to lines DH and KF, respectively, intersecting at E. We then have:

Given $\angle DEF$ (a), with its sides ∥ to and extending in the *same* direction as those of $\angle B$, and $\angle HEK$ (b), with its sides ∥ to and extending in the *opposite* direction to those of $\angle B$.

To Prove $\angle B = \angle a$, and $\angle B = \angle b$.

Proof. 1. Let BC and DH intersect at G; denote $\angle DGC$ by c.

2. Since ∥s AB and DE are cut by BC,

$$\angle B = \angle c.$$

[If two ∥s are cut by a transversal, the corresp. ∠s are equal.] (§ 76)

3. For same reason, since ∥s BC and EF are cut by DE,

$$\angle c = \angle a.$$

4. Then, $\angle B = \angle a$.

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

5. Again, $\angle a = \angle b$.

[If two str. lines intersect, the vertical ∠s are equal.] (§ 44)

6. Then, $\angle B = \angle b$.

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

Note. The sides extend in the *same* direction if they are on the *same* side of a straight line joining the vertices, and in *opposite* directions if they are on *opposite* sides of this line.

82. We have $\angle a$ the supplement of $\angle DEK$. (Fig. of Prop. XIX.)

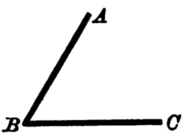
[If two adj. $\angle$s have their ext. sides in the same str. line, their sum is equal to two rt. $\angle$s.] (§ 17)

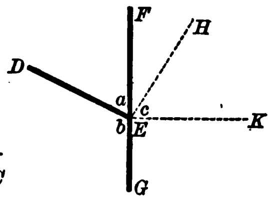
Then its equal, $\angle B$, is the supplement of $\angle DEK$.

That is, *two angles whose sides are parallel, each to each, are supplementary if one pair of parallel sides extend in the same direction, and the other pair in opposite directions, from their vertices.*

Prop. XX. Theorem

83. *Two angles whose sides are perpendicular, each to each, are either equal or supplementary.*





Draw line FG; also, line DE meeting FG at E. Draw lines AB and BC $\perp$ to lines DE and FG, respectively. We then have:

Given $\angle$s DEF (a), DEG (b); $DE \perp AB$, and $FG \perp BC$.

To Prove $\angle B$ equal to $\angle a$, and supplementary to $\angle b$.

Proof. 1. Draw line $EH \perp DE$, and line $EK \perp EF$; denote $\angle HEK$ by c.

2. Since EH and AB are $\perp$ DE, $EH \parallel AB$.

[Two $\perp$s to the same str. line are $\parallel$.] (§ 70)

3. Since EK and BC are $\perp$ FG, $EK \parallel BC$.

4. Then, $\angle c = \angle B$.

[Two $\angle$s whose sides are $\parallel$, each to each, are equal if both pairs of $\parallel$ sides extend in the same direction from their vertices.] (§ 81)

5. Since, by cons., $\angle$s DEH and FEK are rt. $\angle$s, each of the $\angle$s a and c is the complement of $\angle FEH$.

6. Then, $\angle a = \angle c$.

[The complements of equal $\angle$s are equal.] (§ 41)

7. Then, $\angle B = \angle a$.

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

8. Again, $\angle a$ is the supplement of $\angle b$.

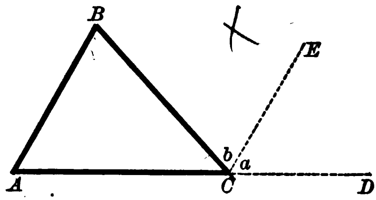
[If two adj. $\angle$s have their ext. sides in the same str. line, they are supplementary.] (§ 17)

9. Then, its equal, $\angle B$, is the supplement of $\angle b$.

Note. The angles are equal if they are both acute or both obtuse; and supplementary if one is acute and the other obtuse.

Prop. XXI. Theorem

84. *The sum of the angles of any triangle is equal to two right angles.*



Given $\triangle ABC$.

To Prove $\angle A + \angle B + \angle C =$ two rt. $\angle$s.

Proof. 1. Extend AC to D, and draw line $CE \parallel AB$; represent $\angle ECD$ by a, and $\angle BCE$ by b.

2. We have $\angle a + \angle b + \angle ACB =$ two rt. $\angle$s. (1)

[The sum of all the $\angle$s on the same side of a str. line at a given point is equal to two rt. $\angle$s.] (§ 18)

3. Since $\parallel$s AB and CE are cut by AD,

$$\angle a = \angle A.$$

[If two $\parallel$s are cut by a transversal, the corresp. $\angle$s are equal.] (§ 76)

4. Since $\parallel$s AB and CE are cut by BC,

$$\angle b = \angle B.$$

[If two $\parallel$s are cut by a transversal, the alt. int. $\angle$s are equal.] (§ 75)

5. Substituting in (1) $\angle A$ for $\angle a$ and $\angle B$ for $\angle b$,

$$\angle A + \angle B + \angle ACB = \text{two rt. } \angle\text{s}.$$

85. We have, from § 84, $\angle BCD = \angle a + \angle b = \angle A + \angle B$.

Hence, *an exterior angle of a triangle is equal to the sum of the two opposite interior angles.*

The following are immediate consequences of §§ 84, 85:

86. *An exterior angle of a triangle is greater than either of the opposite interior angles.*

87. *If two triangles have two angles of one equal, respectively, to two angles of the other, the third angle of the first is equal to the third angle of the second.*

88. *A triangle cannot have two right angles, nor two obtuse angles.*

89. *Two right triangles are equal when a leg and an acute angle of one are equal, respectively, to a leg and the homologous acute angle of the other.* For the remaining acute angles are equal by § 87, and the triangles are equal by § 46.

Ex. 21. Can a triangle be formed whose angles are 35°, 65°, and 55°, respectively? 85°, 50°, and 75°? 50°, 75°, and 55°? 45° 11′ 20″, 61° 52′ 48″, and 72° 55′ 52″?

Ex. 22. The sum of one of the base angles of an isosceles triangle and the vertical angle equals 140°; find the degrees in the exterior angle formed by producing one of the equal sides through the vertex.

Prop. XXII. Theorem

90. (Converse of Prop. IV.) *If two angles of a triangle are equal, the sides opposite are equal.*

Draw $\triangle ABC$ with $\angle A = \angle B$. Represent BC by a and AC by b.

Given, in $\triangle ABC$, $\angle A = \angle B$.

To Prove $a = b$.

The proof is left to the pupil. Draw line $CD \perp AB$. The $\triangle$s ACD and BCD are equal by § 89.

91. From equal $\triangle$s ACD and BCD (Fig. of Prop. XXII),

$$AD = BD, \text{ and } \angle ACD = \angle BCD.$$

Hence, *the perpendicular from the vertex to the base of an isosceles triangle bisects the base and also the vertical angle.*

Prop. XXIII. Theorem

92. *If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle lies opposite the greater side.*

Draw $\triangle ABC$, with $BC\ (a) > AC\ (b)$. **We then have:**

Given $\triangle ABC$ with side $a >$ side b.

To Prove $\angle BAC > \angle B$.

Proof. 1. On CB take $CD = b$, and draw line AD.

2. In isosceles $\triangle ACD$, $\angle CAD = \angle CDA$.

[In an isosceles $\triangle$ the $\angle$s opposite the equal sides are equal.] (§ 50)

3. Since CDA is an ext. $\angle$ of $\triangle ABD$,

$$\angle CDA > \angle B.$$

[An ext. $\angle$ of a $\triangle$ is $>$ either of the opposite int. $\angle$s.] (§ 86)

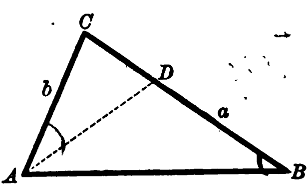
4. Then its equal, $\angle CAD$, is $> \angle B$.

5. Now, $\angle BAC$ is $> \angle CAD$.

6. Then, $\angle BAC$ must be $> \angle B$.

Prop. XXIV. Theorem

93. (Converse of Prop. XXIII.) *If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side lies opposite the greater angle.*



Draw $\triangle ABC$ having $\angle BAC > \angle B$. **We then have:**

Given, in $\triangle ABC$, $\angle BAC > \angle B$.

To Prove $BC\ (a) > AC\ (b)$.

Proof. 1. Draw line AD making $\angle BAD = \angle B$, meeting BC at D.

2. In $\triangle ABD$, $AD = BD$.

[If two $\angle$s of a $\triangle$ are equal, the sides opposite are equal.] (§ 90)

3. But $CD + AD > b$.

[A str. line is the shortest line between two points.] (Ax. 7)

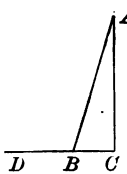
4. Putting for AD its equal BD,

$$CD + BD > b, \text{ or } a > b.$$

The following are immediate consequences of § 93:

94. *The hypotenuse of a right triangle is its greatest side.*

95. *The perpendicular is the shortest line which can be drawn from a point to a straight line.*



For, if AC is the $\perp$, and AB any other line, from A to CD, AB is the hypotenuse of rt. $\triangle ABC$; and hence is $> AC$ (§ 94).

Ex. 23. Two isosceles triangles are equal, if their vertical angles and their bases are respectively equal.

PROP. XXV. THEOREM

96. *Two parallel lines are everywhere equally distant.*

Note. By the *distance* of a point from a line, we mean the length of the perpendicular from the point to the line.

Draw lines AB and $CD \parallel$ (§ 70). At E and F, any two points in AB, draw lines EG and $FH \perp CD$. We then have:

Given $\parallel$s AB and CD, and also EG (a), and FH $(b) \perp CD$.

To Prove $a = b$.

Proof. 1. Draw line FG.

2. We have $a \perp AB$.

[A str. line $\perp$ to one of two $\parallel$s is $\perp$ to the other.] (§ 72)

3. Then, in rt. $\triangle$s EFG and FGH,

$$FG = FG.$$

4. And since ‖s AB and CD are cut by FG,

$$\angle EFG = \angle FGH.$$

[If two ‖s are cut by a transversal, the alt. int. ∠s are equal.] (§ 75)

5. Then $\triangle EFG = \triangle FHG.$

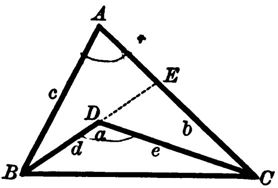
[Two rt. △s are equal when the hypotenuse and an adj. ∠ of one are equal respectively to the hypotenuse and an adj. ∠ of the other.] (§ 60)

6. Then $a = b.$

[In equal figures, the homologous parts are equal.] (§ 48)

Prop. XXVI. Theorem

97. *If straight lines be drawn from a point within a triangle to the extremities of any side, the angle included by them is greater than the angle included by the other two sides.*



Draw any $\triangle ABC$; from any point D within the $\triangle$ draw lines DB and DC. We then have:

Given D, any point within $\triangle ABC$; and lines DB and DC forming $\angle BDC$ (a).

To Prove $\angle a > \angle A.$

Proof. 1. Extend BD to meet AC at E; call $\angle DEC$ b.

2. Since a is an ext. $\angle$ of $\triangle CDE$,

$$\angle a > \angle b.$$

[An ext. ∠ of a △ is > either of the opposite int. ∠s.] (§ 86)

3. Since b is an ext. $\angle$ of $\triangle ABE$,

$$\angle b > \angle A.$$

4. Since $\angle a > \angle b$, and $\angle b > \angle A$, we must have

$$\angle a > \angle A.$$

PROP. XXVII. THEOREM

98. *Any point in the bisector of an angle is equally distant from the sides of the angle.*

Construct any $\angle ABC$. Draw BD, the bisector of $\angle ABC$; from any point P in the bisector, draw lines PM (a) and PN $(b) \perp AB$ and BC, respectively. We then have:

Given P, any point in BD, the bisector of $\angle ABC$, and a and b lines from $P \perp AB$ and BC, respectively.

To Prove $a = b$.

Proof. 1. In rt. ⚠ BPM and BPN, $BP = BP$.

2. By hyp., $\angle PBM = \angle PBN$.

3. Then, $\triangle BPM = \triangle BPN$.

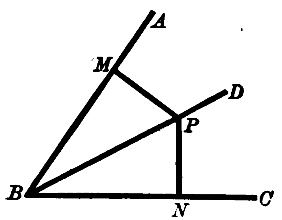
[Two rt. ⚠ are equal when the hypotenuse and an adj. $\angle$ of one are equal respectively to the hypotenuse and an adj. $\angle$ of the other.] (§ 60)

4. Then $PM = PN$.

[In equal figures, the homologous parts are equal.] (§ 48)

PROP. XXVIII. THEOREM

99. (Converse of Prop. XXVII.) *Every point which is within an angle and equally distant from its sides lies in the bisector of the angle.*



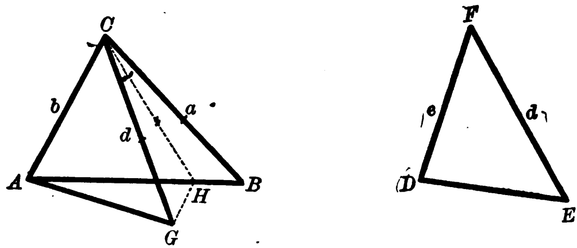
Given point P within $\angle ABC$, equally distant from sides AB and BC, and line BP.

To Prove $\angle PBM = \angle PBN$.

(Prove ⚠ BPM and BPN equal by § 61.)

Prop. XXIX. Theorem

100. *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.*



Draw $\triangle ABC$. Construct $\angle F < \angle ACB$. Draw lines FD (e) and FE (d) equal to CA (b) and CB (a), respectively. Draw line DE. We then have:

Given, in $\triangle$s ABC and DEF, $b = e$, $a = d$, and $\angle ACB > \angle F$.

To Prove $AB > DE$.

Proof. 1. Place $\triangle DEF$ in position ACG; side e coinciding with its equal b, and vertex E falling at G.

2. Draw line CH bisecting $\angle GCB$, meeting AB at H; also, line GH.

3. In $\triangle$s CGH and CBH, $CH = CH$.

4. By hyp., $d = a$, and $\angle GCH = \angle BCH$.

5. Then, $\triangle CGH = \triangle CBH$.

[Two $\triangle$s are equal when two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 46)

6. Then, $GH = BH$.

[In equal figures, the homologous parts are equal.] (§ 48)

7. Now, $AH + GH > AG$.

[A str. line is the shortest line between two points.] (Ax. 7)

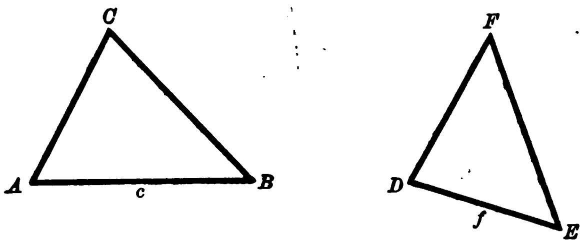
8. Putting for GH its equal BH,

$$AH + BH > AG, \text{ or } AB > DE.$$

Skip

Prop. XXX. Theorem

101. (Converse of Prop. XXIX.) *If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second.*



Draw $\triangle ABC$. Draw $\triangle DEF$ making side $DF = AC$, side $EF = BC$, and side $DE\ (f) < AB\ (c)$. We then have:

Given, in $\triangle$ ABC and DEF, $AC = DF$, $BC = EF$, and $c > f$.

To Prove $\angle C > \angle F$.

Proof. 1. $\angle C$ must be $<$, $=$, or $> \angle F$.

2. If we suppose $\angle C = \angle F$, $\triangle ABC$ would equal $\triangle DEF$.

[Two $\triangle$ are equal when two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 46)

3. Then, c would equal f.

[In equal figures, the homologous parts are equal.] (§ 48)

4. Again, if we suppose $\angle C < \angle F$, c would be $< f$.

[If two $\triangle$ have two sides of one equal respectively to two sides of the other, but the included $\angle$ of the first $>$ the included $\angle$ of the second, the third side of the first is $>$ the third side of the second.] (§ 100)

5. Each of these conclusions is contrary to the hypothesis that c is $> f$.

6. If $\angle C$ cannot be $= \angle F$, nor $< \angle F$, we must have

$$\angle C > \angle F.$$

Ex. 24. ABC is an isosceles triangle. Lines AD and CE, intersecting at O, bisect the equal angles A and C. Prove $\triangle AOE = \triangle DOC$.

Ex. 25. By drawing a line through the vertex of a triangle parallel to the base, prove that the sum of the angles of a triangle is equal to two right angles.

Ex. 26. The line which joins the vertex of an isosceles triangle to the intersection of the bisectors of the exterior angles at the base is a perpendicular bisector of the base.

Ex. 27. If one of the equal sides of an isosceles triangle be extended through the vertex, a line through the vertex parallel to the base is a bisector of the exterior angle at the vertex.

Ex. 28. The bisectors of the base angles of an isosceles triangle meet in a point and form with the base an isosceles triangle.

Ex. 29. Through the middle point of one of the equal sides of an isosceles triangle a line parallel to the other equal side is drawn to meet the base and the bisector of the exterior angle at the vertex. Prove the two triangles formed equal.

Ex. 30. The bisectors of the base angles of an isosceles triangle form an angle of 110°. Find its vertical angle.

Ex. 31. Prove Prop. XXIX when point E falls *within* triangle ABC.

Ex. 32. The line which joins the vertex of an isosceles triangle to the intersection of the bisectors of the base angles, bisects the vertical angle.

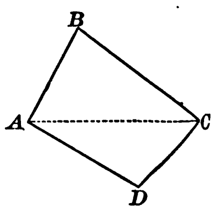
Ex. 33. The line which bisects the vertical angle of an isosceles triangle bisects the base at right angles.

Ex. 34. Is it always possible to find a point in a given line which is equidistant from two given intersecting lines? Is there ever more than one such point?

QUADRILATERALS

DEFINITIONS

102. A *quadrilateral* is a portion of a plane bounded by four straight lines; as $ABCD$.



The bounding lines are called the *sides* of the quadrilateral, and their points of intersection the *vertices*.

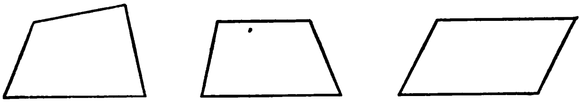
The *angles* of the quadrilateral are the angles formed by the adjacent sides, ABC, BCD, etc.

A *diagonal* is a straight line joining two opposite vertices; as AC.

103. A *Trapezium* is a quadrilateral no two of whose sides are parallel.

A *Trapezoid* is a quadrilateral two, and only two, of whose sides are parallel.

A *Parallelogram* is a quadrilateral whose opposite sides are parallel.



Trapezium Trapezoid Parallelogram

An *isosceles* trapezoid is a trapezoid whose non-parallel sides are equal.

The *bases* of a trapezoid are its parallel sides; the *altitude* is the perpendicular distance between them.

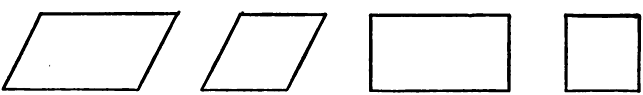
If either pair of parallel sides of a parallelogram be taken and called the *bases*, the *altitude* corresponding to these bases is the perpendicular distance between them.

A *Rhomboid* is a parallelogram. whose angles are not right angles, and whose adjacent sides are unequal.

A *Rhombus* is a parellelogram whose angles are not right angles, and whose adjacent sides are equal.

A *Rectangle* is a parallelogram whose angles are right angles.

A *Square* is a rectangle whose sides are equal.

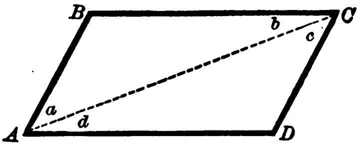


Rhomboid Rhombus Rectangle Square

Ex. 35. Bisect the exterior angles of a given triangle and join the vertices of the triangle thus formed to the opposite vertices of the given triangle. The lines thus drawn are the altitudes of the triangle formed by the bisectors.

Prop. XXXI. Theorem

104. *In any parallelogram, the opposite sides are equal, and the opposite angles are equal.*



Draw the figure in accordance with the statement of the proposition. We then have:

Given ▱ $ABCD$.

To Prove

$$AB = CD,\ BC = AD,\ \angle B = \angle D, \text{ and } \angle BAD = \angle BCD.$$

Proof. 1. Draw diagonal AC; denote ∠s BAC, BCA, ACD, and CAD by a, b, c, and d, respectively.

2. In △s ABC and ACD, $AC = AC$.

3. Since ∥s AB and CD are cut by AC,

$$\angle a = \angle c. \tag{1}$$

[If two ∥s are cut by a transversal, the alt. int. ∠s are equal.] (§ 75)

4. Since ∥s BC and AD are cut by AC,

$$\angle b = \angle d. \tag{2}$$

5. Then, $\triangle ABC = \triangle ACD$.

[Two △s are equal when a side and two adj. ∠s of one are equal respectively to a side and two adj. ∠s of the other.] (§ 49)

6. Then, $AB = CD$, $BC = AD$, and $\angle B = \angle D$.

[In equal figures, the homologous parts are equal.] (§ 48)
(Why are AB and CD, and BC and AD, homologous lines?)

7. Adding (1) and (2), we have

$$\angle a + \angle b = \angle c + \angle d, \text{ or } \angle BAD = \angle BCD.$$

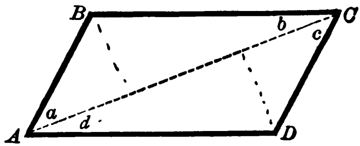
The following are consequences of § 104:

105. 1. *Parallel lines included between parallel lines are equal.*

2. *A diagonal of a parallelogram divides it into two equal triangles.*

PROP. XXXII. THEOREM

106. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Draw quadrilateral $ABCD$ having side $BC = AD$, and side $AB = DC$. We then have:

Given, in quadrilateral $ABCD$, $AB = CD$ and $BC = AD$.

To Prove $ABCD$ a $\square$.

Proof. 1. Draw diagonal AC; represent $\angle s$ BAC, BCA, ACD, and CAD by a, b, c, and d, respectively.

2. In $\triangle s$ ABC and ACD, $AC = AC$.

3. By hyp., $AB = CD$ and $BC = AD$.

4. Then, $\triangle ABC = \triangle ACD$.

[Two $\triangle s$ are equal when the three sides of one are equal respectively to the three sides of the other.] (§ 53)

5. Then, $\angle a = \angle c$, and $\angle b = \angle d$.

[In equal figures, the homologous parts are equal.] (§ 48)

6. Since $\angle a = \angle c$, $AB \parallel CD$.

[If two str. lines are cut by a transversal, and the alt. int. $\angle s$ are equal, the two lines are $\parallel$.] (§ 78)

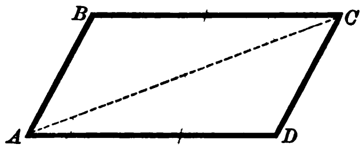
7. Since $\angle b = \angle d$, $BC \parallel AD$.

8. Then, by def., $ABCD$ is a $\square$.

Ex. 36. The perpendiculars from two opposite vertices of a parallelogram to a diagonal are equal.

Prop. XXXIII. Theorem

107. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Draw line $BC =$ and $\parallel$ to AD, and lines AB and DC. We then have:

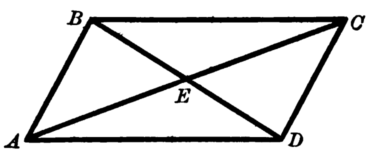
Given, in quadrilateral $ABCD$, BC equal and $\parallel$ to AD.

To Prove $ABCD$ a ▱.

(Prove $\triangle$s ABC and ACD equal by § 46; then, $AB = CD$. Compare § 106.)

Prop. XXXIV. Theorem

108. *The diagonals of a parallelogram bisect each other.*



Draw ▱ $ABCD$ as in § 107; and lines AC and BD. We then have:

Given diagonals AC and BD of ▱ $ABCD$ intersecting at E.

To Prove $AE = EC$ and $BE = ED$.

(Prove $\triangle AED = \triangle BEC$, by § 49.)

Note. The point E is called the *centre* of the parallelogram.

Prop. XXXV. Theorem

109. (Converse of Prop. XXXIII.) *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

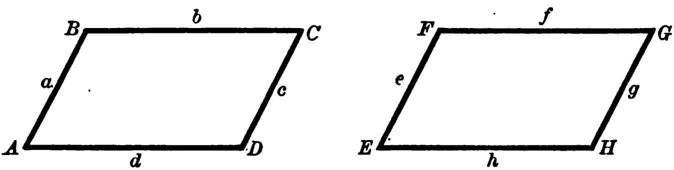
Given AC and BD, the diagonals of quadrilateral $ABCD$, bisecting each other at E. (Fig. of Prop. XXXIV.)

To Prove $ABCD$ a ▱.

(Prove $\triangle AED = \triangle BEC$, by § 46; then $AD = BC$; in like manner, $AB = CD$; then use § 106.)

PROP. XXXVI. THEOREM

110. *Two parallelograms are equal when two adjacent sides and the included angle of one are equal respectively to two adjacent sides and the included angle of the other.*



Draw ▱s $ABCD$ and $EFGH$ in accordance with the statement of the proposition. Let a, b, c, d, represent sides AB, BC, CD, and DA, respectively; and e, f, g, h, sides EF, FG, GH, and HE, respectively. We then have:

Given, in ▱s $ABCD$ and $EFGH$,

$$a = e,\ d = h, \text{ and } \angle A = \angle E.$$

To Prove ▱ $ABCD$ = ▱ $EFGH$.

Proof. 1. Superpose ▱ $ABCD$ upon ▱ $EFGH$ in such a way that $\angle A$ shall coincide with its equal $\angle E$; side a falling on side e, and side d on side h.

2. Since $a = e$, point B will fall on point F.

3. Since $d = h$, point D will fall on point H.

4. Since $b \parallel d$, and $f \parallel h$, side b will fall on side f, and point C will fall somewhere on f.

[But one str. line can be drawn through a given point ∥ to a given str. line.] (§ 69)

5. Since $c \parallel a$, and $g \parallel e$, side c will fall on side g, and point C will fall somewhere on g.

6. Since C falls on both f and g, it must fall at their intersection, G.

7. Then, the ▱s coincide throughout, and are equal.

The following is an immediate consequence of § 110:

111. *Two rectangles are equal if the base and altitude of one are equal respectively to the base and altitude of the other.*

PROP. XXXVII. THEOREM

112. *The diagonals of a rectangle are equal.*

Draw rectangle $ABCD$; draw lines AC and BD. We then have:

Given AC and BD the diagonals of rect. $ABCD$.

To Prove $AC = BD$.

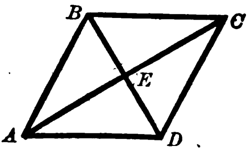
(Prove rt. $\triangle ABD =$ rt. $\triangle ACD$, by § 46.)

The following is an immediate consequence of § 112:

113. *The diagonals of a square are equal.*

PROP. XXXVIII. THEOREM

114. *The diagonals of a rhombus bisect each other at right angles.*



Given AC and BD the diagonals of rhombus $ABCD$.

To Prove that AC and BD bisect each other at rt. $\angle$s. (Compare § 56.)

Ex. 37. The lines which join the middle points of the opposite sides of a parallelogram bisect each other.

Ex. 38. Are the diagonals of a parallelogram ever equal?

Ex. 39. When do the diagonals of a parallelogram bisect the opposite angles?

Ex. 40. If perpendiculars be drawn from the extremities of the upper base of an isosceles trapezoid to the lower base, they are equal and cut off equal segments from the extremities of the lower base.

Ex. 41. The lines which join the middle points of the bases of an isosceles trapezoid to the middle points of the non-parallel sides form, with the sides of the trapezoid, two pairs of equal triangles.

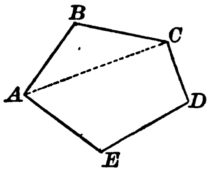
Ex. 42. If lines be drawn from the middle point of the base of an isosceles triangle to the middle points of the equal sides, the triangle is divided into two equal triangles and a parallelogram.

Learn

POLYGONS

DEFINITIONS

115. We define a *polygon* as a portion of a plane bounded by three or more straight lines; as $ABCDE$.



We call the bounding lines the *sides* of the polygon, and their sum the *perimeter*.

The *angles* of the polygon are the angles EAB, ABC, etc., formed by the adjacent sides; their vertices are called the *vertices* of the polygon.

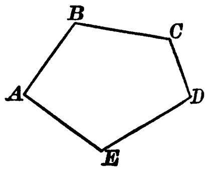
A *diagonal* of a polygon is a straight line joining any two vertices which are not consecutive; as AC.

116. Polygons are named with reference to the number of their sides, as follows:

No. of Sides.	Designation.	No. of Sides.	Designation.
3	Triangle.	8	Octagon.
4	Quadrilateral.	9	Enneagon.
5	Pentagon.	10	Decagon.
6	Hexagon.	11	Hendecagon.
7	Heptagon.	12	Dodecagon.

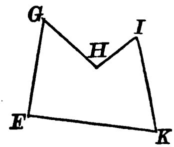
117. An *equilateral* polygon is a polygon all of whose sides are equal.

An *equiangular* polygon is a polygon all of whose angles are equal.



118. A polygon is called *convex* when no side, if extended, will enter its surface; as $ABCDE$; in such a case, each angle of the polygon is less than two right angles.

Every polygon considered hereafter will be understood to be convex, unless the contrary is stated.

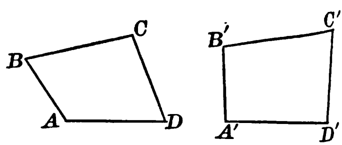


119. A polygon is called *concave* when at least two of its sides, if extended, will enter its surface; as $FGHIK$.

In such a case, at least one angle of the polygon is greater than two right angles.

Thus, in polygon $FGHIK$, the interior angle GHI is greater than two right angles; such an angle is called *re-entrant*.

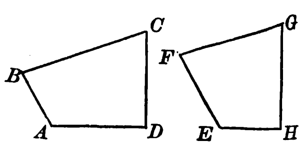
120. We call two polygons *mutually equilateral* when the sides of one are equal respectively to the sides of the other, when taken in the same order.



Thus, polygons $ABCD$ and $A'B'C'D'$ are mutually equilateral if

$AB = A'B'$, $BC = B'C'$, $CD = C'D'$, and $DA = D'A'$.

121. We call two polygons *mutually equiangular* when the angles of one are equal respectively to the angles of the other, when taken in the same order.



Thus, polygons $EFGH$ and $E'F'G'H'$ are mutually equiangular if

$$\angle A = \angle E,\ \angle B = \angle F,\ \angle C = \angle G, \text{ and } \angle D = \angle H.$$

122. In polygons which are mutually equilateral or mutually equiangular, sides or angles which are similarly placed are called *homologous*.

123. *If two polygons are both mutually equilateral and mutually equiangular, they are equal.*

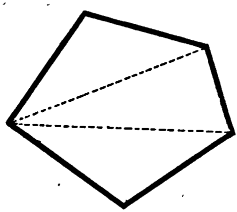
For they can evidently be applied one to the other so as to coincide throughout.

124. *Two polygons are equal when they are composed of the same number of triangles, equal each to each, and similarly placed.*

For they can evidently be applied one to the other so as to coincide throughout.

PROP. XXXIX. THEOREM

125. *The sum of the angles of any polygon is equal to two right angles taken as many times, less two, as the polygon has sides.*



Draw any polygon. We then have:

Given a polygon of n sides.

To Prove the sum of its $\angle$s equal to $n-2$ times two rt. $\angle$s.

Proof. 1. The polygon may be divided into $n-2$ $\triangle$s by drawing diagonals from one of its vertices.

2. The sum of the $\angle$s of the polygon is equal to the sum of the $\angle$s of the $\triangle$s.

3. Now the sum of the $\angle$s of each $\triangle$ is two rt. $\angle$s.

[The sum of the $\angle$s of any $\triangle$ is equal to two rt. $\angle$s.] (§ 84)

4. Then, the sum of the $\angle$s of the polygon is $n-2$ times two rt. $\angle$s.

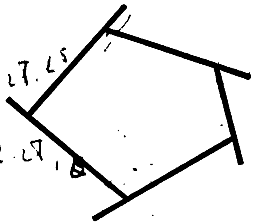
126. We have $(n-2) \times 2$ rt. $\angle$s equal to $2n$ rt. $\angle$s $- 4$ rt. $\angle$s.

Then, *the sum of the angles of any polygon equals twice as many right angles as the polygon has sides, less four right angles.*

Prop. XL. Theorem

127. *If the sides of any polygon be produced so as to form an exterior angle at each vertex, the sum of these exterior angles is equal to four right angles.* 360

$(n-2) \times 2$ rt. ∠s



Draw a polygon in accordance with the statement of the proposition. We then have:

Given a polygon of n sides with its sides extended so as to form an ext. ∠ at each vertex.

To Prove the sum of the ext. ∠s equal to 4 rt. ∠s.

Proof. 1. The sum of the ext. and int. ∠s at each vertex is two rt. ∠s.

[If two adj. ∠s have their ext. sides in the same str. line, their sum is equal to two rt. ∠s.] (§ 17)

2. Hence, the sum of *all* the ext. and int. ∠s is $2n$ rt. ∠s.

3. But the sum of the int. ∠s alone is $2n$ rt. ∠s − 4 rt. ∠s.

[The sum of the ∠s of any polygon equals twice as many rt. ∠s as the polygon has sides, less 4 rt. ∠s.] (§ 126)

4. Whence, the sum of the ext. ∠s is 4 rt. ∠s.

4 rt. ∠s 6 rt. ∠s

Ex. 43. Find the sum of the angles of a quadrilateral; of a pentagon; of a hexagon; of an octagon.

Ex. 44. An exterior angle of an equiangular polygon is one-fifth of a right angle. How many sides has the polygon?

Ex. 45. How many sides has a polygon each of whose interior angles is eight-fifths of a right angle?

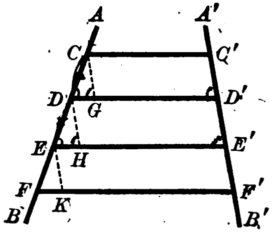
Ex. 46. The difference between two consecutive angles of a parallelogram is 75°; find the angles of the parallelogram. Find the angles when this difference is $M°$.

Ex. 47. How many sides has the polygon the sum of whose interior angles is 14 right angles? Find the magnitude of each angle if the polygon is equiangular.

MISCELLANEOUS THEOREMS

Prop. XLI. Theorem

128. *If a series of parallels, cutting two straight lines, intercept equal distances on one of these lines, they also intercept equal distances on the other.*



Draw a series of parallels (§ 70). Draw lines AB and $A'B'$ cutting these ‖s at C, D, E, F, and C', D', E', F', respectively. We then have:

Given lines AB and $A'B'$ cut by ‖s CC', DD', EE', and FF' at points C, D, E, F, and C', D', E', F', respectively, so that

$$CD = DE = EF.$$

To Prove $\quad C'D' = D'E' = E'F'.$

Proof. 1. Draw lines CG, DH, and EK ‖ $A'B'$, meeting DD', EE', and FF' at points G, H, and K, respectively.

2. In $\triangle$s CDG, DEH, EFK, by hyp., $CD = DE = EF$.

3. Again, lines CG, DH, and EK are ‖ to each other.

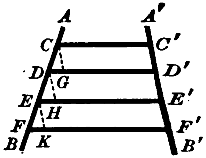
[Two str. lines ‖ to the same str. line are ‖ to each other.] (§ 71)

4. Then, $\angle DCG = \angle EDH = \angle FEK$.

[If two ‖ lines are cut by a transversal, the corresp. $\angle$s are equal.] (§ 76)

5. Also, $\angle CDG = \angle DEH = \angle EFK$.

[If two ‖ lines are cut by a transversal, the corresp. $\angle$s are equal.] (§ 76)



6. Then, $\triangle CDG = \triangle DEH = \triangle EFK$.

[Two $\triangle$ are equal when a side and two adj. $\angle$ of one are equal respectively to a side and two adj. $\angle$ of the other.] (§ 49)

7. Then, $CG = DH = EK$.

[In equal figures, the homologous parts are equal.] (§ 48)

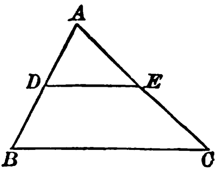
8. But, $CG = C'D'$, $DH = D'E'$, and $EK = E'F'$.

[In any $\square$, the opposite sides are equal.] (§ 104)

9. Then, $C'D' = D'E' = E'F'$.

[Things which are equal to equal things, are equal to each other.] (Ax. 1)

129. *The line which bisects one side of a triangle, and is parallel to another side, bisects also the third side.*



If line DE is $\parallel$ side BC of $\triangle ABC$, and bisects side AB, it also bisects AC.

This is the particular case in the figure of Prop. XLI where CG coincides with $C'D'$.

130. In the figure of Prop. XLI, $CC'E'E$ is a trapezoid, and DD' bisects its sides CE and $C'E'$.

Hence, *the line which is parallel to the bases of a trapezoid and bisects one of the non-parallel sides, bisects the other also.*

Prop. XLII. Theorem

131. *The line joining the middle points of two sides of a triangle is parallel to the third side, and equal to one-half of it.*

Draw any $\triangle ABC$. Through D and E, middle points of AB and AC, respectively, draw line DE. We then have:

Given line DE joining middle points of sides AB and AC, respectively, of $\triangle ABC$.

To Prove $DE \parallel BC$, and $DE = \frac{1}{2} BC$.

Proof. 1. A line from $D \parallel BC$ will bisect AC.

[The line which bisects one side of a $\triangle$, and is $\parallel$ to another side, bisects also the third side.] (§ 129)

2. Then it must coincide with DE, and $DE \parallel BC$.

[But one str. line can be drawn between two points.] (Ax. 6)

3. Draw line $EF \parallel AB$, meeting side BC at F.
4. In $\triangle$s ADE and EFC, by hyp., $AE = EC$.
5. Also, $\angle A = \angle CEF$, and $\angle AED = \angle C$.

[If two ∥s are cut by a transversal, the corresp. ∠s are equal.] (§ 76)

6. Then, $\triangle ADE = \triangle EFC$.

[Two $\triangle$s are equal when they have a side and two adjacent ∠s of one equal respectively to a side and two adjacent ∠s of the other.] (§ 49)

7. Then, $DE = FC$.

[In equal figures, the homologous parts are equal.] (§ 48)

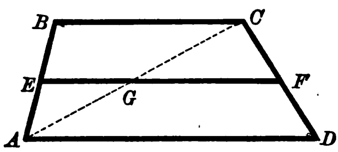
8. But, $DE = BF$.

[In any ▱, the opposite sides are equal.] (§ 104)

9. Then, $DE = \frac{1}{2} BC$.

Prop. XLIII. Theorem

132. *The line joining the middle points of the non-parallel sides of a trapezoid is parallel to the bases, and equal to one-half their sum.*



Draw trapezoid $ABCD$, BC and AD being ∥ sides. Let E, F be middle points of AB and CD, respectively. Draw line EF. We then have:

Given line EF joining middle points of non-∥ sides AB and CD, respectively, of trapezoid $ABCD$.

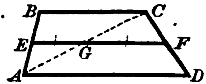
To Prove $EF \parallel$ to AD and BC, and $EF = \frac{1}{2}(AD + BC)$.

Proof. 1. A line from $E \parallel$ to AD and BC will bisect CD.

[The line which is ∥ to the bases of a trapezoid, and bisects one of the non-∥ sides, bisects the other also.] (§ 130)

2. Then it coincides with EF, and EF is $\parallel$ to AD and BC.

[But one str. line can be drawn between two points.] (Ax. 6)



3. Draw diagonal AC, intersecting EF at G.

4. Since EG bisects side AB of $\triangle ABC$, and is $\parallel$ BC, it bisects side AC.

[The line which bisects one side of a $\triangle$, and is $\parallel$ to another side, bisects also the third side.] (§ 129)

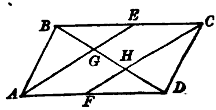
5. Then, $EG = \frac{1}{2} BC$. (1)

[The line joining the middle points of two sides of a $\triangle$ is $\parallel$ to the third side, and equal to one-half of it.] (§ 131)

6. In like manner, $GF = \frac{1}{2} AD$. (2)

7. Adding (1) and (2), $EF = \frac{1}{2} BC + \frac{1}{2} AD = \frac{1}{2}(BC + AD)$.

Ex. 48. A straight line is drawn from the vertex of a triangle to any point in the base. Prove that this line is bisected by the line joining the middle points of the sides of the triangle.



Ex. 49. E and F are middle points of sides BC and AD, respectively, of the parallelogram $ABCD$. Draw lines AE and CF. Prove that these lines trisect the diagonal joining the other two vertices of the parallelogram.

Prop. XLIV. Theorem

133. *The bisectors of the angles of a triangle intersect at a common point.*

Draw any $\triangle ABC$. Let a, b, c, be the bisectors of $\angle$s A, B, and C, respectively. We then have:

Given a, b, and c, the bisectors of the angles of $\triangle ABC$.

To Prove that a, b, and c intersect at a common point.

Proof. 1. Let a and b intersect at O.

2. Since O is in bisector a, it is equally distant from sides AB and AC.

[Any point in the bisector of an $\angle$ is equally distant from the sides of the $\angle$.] (§ 98)

3. Since O is in bisector b, it is equally distant from sides AB and BC.

4. Then O is equally distant from sides AC and BC, and therefore lies in bisector c.

[Every point which is within an $\angle$, and equally distant from its sides, lies in the bisector of the $\angle$.] (§ 99)

5. Then, a, b, and c intersect in the common point O.

134. It follows from § 133 that

The point of intersection of the bisectors of the angles of a triangle is equally distant from the sides of the triangle.

Ex. 50. The lines joining the middle points of the bases of an isosceles trapezoid to the middle points of the non-parallel sides form a rhombus.

Ex. 51. If through the vertex A, of isosceles triangle ABC, a parallel to BC be drawn, and through the middle points of AB and AC parallels, respectively, to AC and AB, a triangle is formed equal to the given triangle, and its vertex will lie in the base of the given triangle.

Ex. 52. Two sides of a quadrilateral are parallel. The other sides are equal, but not parallel. Prove the opposite angles supplementary.

Prop. XLV. Theorem

135. *The perpendiculars erected at the middle points of the sides of a triangle intersect at a common point.*

Draw any $\triangle ABC$; at D, E, and F, middle points of BC, CA, and AB, respectively, of $\triangle ABC$, draw lines DG, EH, and $FK \perp$ to BC, CA, and AB, respectively. We now have:

Given DG, EH, and FK, the $\perp$s erected at the middle points of the sides of $\triangle ABC$.

To Prove that DG, EH, and FK intersect at a common point.

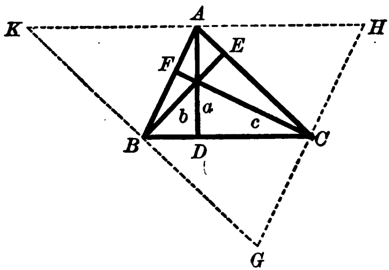
(If DG and EH intersect at O, O is equally distant from B and C, by § 54; and also from A and C.)

136. It follows from § 135 that

The point of intersection of the perpendiculars erected at the middle points of the sides of a triangle, is equally distant from the vertices of the triangle.

Prop. XLVI. Theorem

137. *The perpendiculars from the vertices of a triangle to the opposite sides intersect at a common point.*



Draw any $\triangle ABC$. From A, B, and C draw lines AD (a), BE (b), and CF (c) $\perp$ to BC, CA, and AB, respectively. We then have:

Given a, b, and c, the three $\perp$s from the vertices of $\triangle ABC$ to the opposite sides.

To Prove that a, b, and c intersect at a common point.

Proof. 1. Through A, B, and C draw lines HK, KG, and $GH \parallel BC$, CA, and AB, respectively, forming $\triangle GHK$.

2. Since a is $\perp BC$, it is also $\perp HK$.

[A str. line $\perp$ to one of two $\parallel$s is $\perp$ to the other.] (§ 72)

3. Since, by cons., $ABCH$ and $ACBK$ are $\boxed{s}$,

$$AH = BC \text{ and } AK = BC.$$

[In any $\square$, the opposite sides are equal.] (§ 104)

4. Then, $AH = AK$.

[Things which are equal to the same thing, are equal to each other.] (Ax. 1)

5. Then, a is $\perp HK$ at its middle point.

6. In like manner, b and c are $\perp$ to KG and GH, respectively, at their middle points.

7. Since a, b, and c are $\perp$ to the sides of $\triangle GHK$ at their middle points, they intersect at a common point.

[The $\perp$s erected at the middle points of the sides of a $\triangle$ intersect at a common point.] (§ 135)

138. Def. A *median* of a triangle is a line drawn from any vertex to the middle point of the opposite side.

PROP. XLVII. THEOREM

139. *The line drawn from any vertex of a triangle through the point of intersection of medians drawn from the other two vertices, is itself a median.*

Draw $\triangle ABC$; draw medians BE and AD intersecting at O; through O draw line CO meeting AB at F. We then have:

Given AD and BE, two medians of $\triangle ABC$ intersecting at O.

To Prove CF also a median, or $AF = BF$.

Proof. 1. Draw line $AG \parallel BE$, meeting CF prolonged at G; also, line BG.

2. Since EO is $\parallel$ to side AG of $\triangle ACG$, and bisects side CA, O is the middle point of CG.

[The line which bisects one side of a $\triangle$, and is $\parallel$ to another side, bisects also the third side.] (§ 129)

3. Since O is the middle point of CG, and D of CB, OD is $\parallel$ to side BG of $\triangle BCG$.

[The line joining the middle points of two sides of a $\triangle$ is $\parallel$ to the third side.] (§ 131)

4. Since $AG \parallel BE$, and $AO \parallel BG$, $AOBG$ is a $\square$, and $AF = BF$.

[The diagonals of a $\square$ bisect each other.] (§ 108)

140. In the figure of Prop. XLVII, $OF = FG$. (§ 108)

Since O is the middle point of CG, $OG = OC$.

Then, $OF = \frac{1}{2} OG = \frac{1}{2} OC$; and hence $OC = \frac{2}{3} CF$.

In like manner, $OA = \frac{2}{3} AD$, and $OB = \frac{2}{3} BE$.

That is, *the point of intersection of the medians of a triangle lies two-thirds the way from each vertex to the opposite side.*

141. As an aid in the solution of original exercises, the following list of certain principles proved in Book I will be found useful.

Two triangles are equal :

When two sides and the included angle of one are equal respectively to two sides and the included angle of the other (§ 46).

When a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other (§ 49).

When the three sides of one are equal respectively to the three sides of the other (§ 52).

Two right triangles are equal :

When the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other (§ 60).

When the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other (§ 61).

When a leg and an acute angle of one are equal respectively to a leg and the homologous acute angle of the other (§ 89).

Two straight lines are equal :

When they are homologous sides of equal triangles (§ 48).

When they are opposite equal angles in a triangle (§ 90).

When they are opposite sides of a parallelogram (§ 104).

When they are diagonals of a rectangle (§ 112).

Two angles are equal :

When they are the complements or supplements of equal angles (§ 41).

When they are vertical (§ 44).

When they are homologous angles of equal triangles (§ 48).

When they are opposite the equal sides of an isosceles triangle (§ 50).

When they are alternate-interior angles (§ 75).

When they are corresponding angles (§ 76).

When their sides are parallel each to each (§ 81).

When their sides are perpendicular each to each (§ 83).

When they are opposite angles of a parallelogram (§ 104).

Two lines are parallel :

When they are perpendicular to the same line (§ 70).

When they are parallel to the same line (§ 71).

When the alternate-interior angles are equal (§ 78).

When the corresponding angles are equal (§ 79).

When the sum of the interior angles on the same side of the transversal is equal to two right angles (§ 80).

When they are opposite sides of a parallelogram.

When one line is a side of a triangle, and the other joins the middle points of the other two sides (§ 131).

A quadrilateral is a parallelogram :

When the opposite sides are equal (§ 106).
When two sides are equal and parallel (§ 107).
When the diagonals bisect each other (§ 109).

One line is greater than another :

When it is the hypotenuse of a right triangle of which the other line is a leg (§ 94).
When it is opposite a greater angle in a triangle (§ 93).

One angle is greater than another :

When it is an exterior angle of a triangle in which the other is an opposite interior angle (§ 86).
When it is opposite a greater side in a triangle (§ 92).

Ex. 53. Is it always possible to find a point equidistant from three given straight lines?

Ex. 54. Is it possible to find a point equidistant from four given straight lines?

Ex. 55. If two medians of a triangle intersect the sides at right angles, the triangle is equiangular.

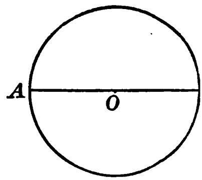
Ex. 56. If a line joining two opposite vertices of a parallelogram bisects the angles at these vertices, the parallelogram is equilateral.

Ex. 57. Prove that lines drawn from two vertices of a triangle and terminating in the opposite sides cannot bisect each other.

Book II

THE CIRCLE

DEFINITIONS



142. A *diameter* of a circle is a straight line drawn through the centre, having its extremities in the circumference.

143. By the definition of § 23

All radii of a circle are equal.

Also, all its diameters are equal, since each is the sum of two radii.

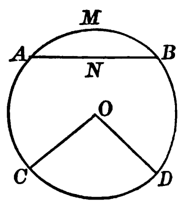
144. *Two circles are equal when their radii are equal.*

For they can evidently be applied one to the other so that their circumferences shall coincide throughout.

145. *The radii of equal circles are equal.* (Converse of § 144.)

146. *Concentric circles* are circles having the same centre.

147. A *chord* is a straight line joining the extremities of an arc; as *AB*.



The arc is said to be *subtended* by its chord.

Every chord subtends two arcs; thus chord *AB* subtends arcs *AMB* and *ACDB*.

When *the* arc subtended by a chord is spoken of, the one which is less than a semicircumference is understood, unless the contrary is specified.

A *segment* of a circle is the portion included between an arc and its chord; as *AMBN*.

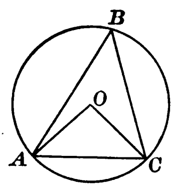
A *sector* of a circle is the portion included between an arc and the radii drawn to its extremities; as *OCD*.

148. *A straight line cannot meet a circumference at more than two points.*

For by § 64, not more than two equal straight lines can be drawn from a point (in this case, the centre) to a straight line.

149. A *central angle* is an angle whose vertex is at the centre, and whose sides are radii; as *AOC*.

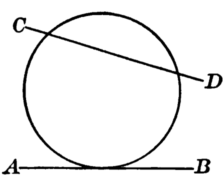
An *inscribed angle* is an angle whose vertex is on the circumference, and whose sides are chords; as *ABC*.



An angle is *inscribed in a segment* when its vertex is on the arc of the segment, and its sides pass through the extremities of the subtending chord; thus, angle *B* is inscribed in segment *ABC*.

150. A straight line is said to be *tangent to*, or to *touch, a circle* when it has but one point in common with the circumference; as *AB*.

In such a case, the circle is said to be tangent to the straight line.



The common point is called the *point of contact*.

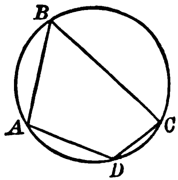
A *secant* is a straight line which intersects the circumference in two points; as *CD*.

151. Two circles are said to be *tangent to each other* when they are tangent to the same straight line at the same point.

They are said to be tangent *internally* or *externally* according as one circle lies entirely within or entirely without the other.

A *common tangent* to two circles is a straight line which is tangent to both of them.

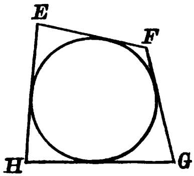
152. A polygon is said to be *inscribed in a circle* when all its vertices lie on the circumference; as $ABCD$.



In this case, the circle is said to be *circumscribed about the polygon.*

A polygon is said to be *inscriptible* when it can be inscribed in a circle.

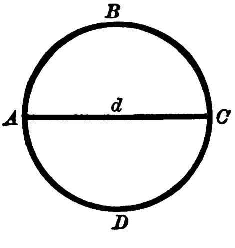
A polygon is said to be *circumscribed about a circle* when all its sides are tangent to the circle; as $EFGH$.



In this case, the circle is said to be *inscribed in the polygon.*

Prop. I. Theorem

153. *Every diameter bisects the circle and its circumference.*



Given $AC(d)$ a diameter of $\odot$ $ABCD$.

To Prove that d bisects the $\odot$, and its circumference.

Proof. 1. Superpose segment ABC upon segment ADC, by folding it over about d as an axis.

2. Then, arc ABC will coincide with arc ADC; for otherwise there would be points of the circumference unequally distant from the centre.

3. Hence, segments ABC and ADC coincide throughout, and are equal.

4. Therefore, d bisects the $\odot$, and its circumference.

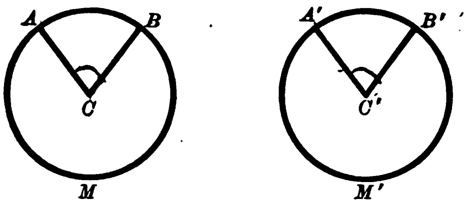
154. Defs. A *semi-circumference* is an arc equal to one-half the circumference.

A *quadrant* is an arc equal to one-fourth the circumference.
A *semicircle* is a segment equal to one-half the circle.

Ex. 1. If two non-perpendicular diameters be drawn in the same circle, the perpendiculars from the extremities of the one upon the other are equal.

PROP. II. THEOREM

155. *In equal circles, or in the same circle, equal central angles intercept equal arcs on the circumference.*



Draw ⊙ *AMB* and *A'M'B'*, having centres *C* and *C'*, respectively, and same radius. In ⊙ *AMB*, draw ∠ *ACB*; in ⊙ *A'M'B'*, construct ∠ *A'C'B'* = ∠ *ACB*. We then have:

Given *ACB* and *A'C'B'* equal central ∠s of equal ⊙ *AMB* and *A'M'B'*, respectively.

To Prove arc *AB* = arc *A'B'*.

Proof. 1. Superpose sector *ABC* upon sector *A'B'C'* in such a way that ∠ *C* shall coincide with its equal ∠ *C'*.

2. We have $AC = A'C'$ and $BC = B'C'$. (§ 145)

3. Then, point *A* will fall at *A'*, and point *B* at *B'*.

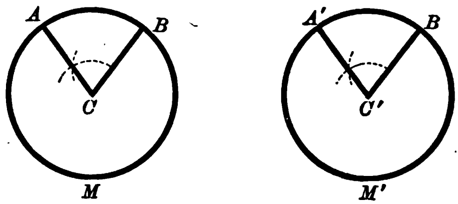
4. Then, arc *AB* will coincide with arc *A'B'*; for all points of either are equally distant from the centre.

5. Whence, arc *AB* = arc *A'B'*.

Ex. 2. Draw lines dividing a circumference into eight equal parts.

Prop. III. Theorem

156. (Converse of Prop. II.) *In equal circles, or in the same circle, equal arcs are intercepted by equal central angles.*



Draw Ⓢ AMB and $A'M'B'$, having centres C and C', respectively, and same radius. Take arc AB = arc $A'B'$, and draw lines AC, BC, $A'C'$, and $B'C'$. We then have:

Given ACB and $A'C'B'$ central $\angle$s of equal Ⓢ AMB and $A'M'B'$, respectively, and arc AB = arc $A'B'$.

To Prove $\angle C = \angle C'$.

Proof. 1. Since the Ⓢ are equal, we may superpose $\odot$ AMB upon $\odot$ $A'M'B'$ in such a way that point A shall fall at A', and centre C at C'.

2. Since arc AB = arc $A'B'$, point B will fall at B'.

3. Then, radii AC and BC will coincide with radii $A'C'$ and $B'C'$, respectively. (Ax. 6)

4. Then, $\angle C$ will coincide with $\angle C'$; that is, $\angle C = \angle C'$.

157. Note. *In equal circles, or in the same circle,*

1. *The greater of two central angles intercepts the greater arc on the circumference.*

2. *The greater of two arcs is intercepted by the greater central angle.*

Prop. IV. Theorem

158. *In equal circles, or in the same circle, equal chords subtend equal arcs.*

Draw Ⓢ AMB and $A'M'B'$, having same radius. In $\odot$ AMB, draw chord AB; in $\odot$ $A'M'B'$, draw chord $A'B'$ = chord AB. We then have:

Given, in equal Ⓢ AMB, $A'M'B'$, chord AB = chord $A'B'$.

To Prove arc $AB =$ arc $A'B'$.

Proof. 1. Draw radii AC, BC, $A'C'$, and $B'C'$.

2. In $\triangle$ ABC and $A'B'C'$, by hyp., $AB = A'B'$.

3. Also, $AC = A'C'$, and $BC = B'C'$. (§ 145)

4. Then, $\triangle ABC = \triangle A'B'C'$. (§ 52)

5. Then, $\angle C = \angle C'$. (?)

6. Then, arc $AB =$ arc $A'B'$. (§ 155)

Ex. 3. If A, B, C, D are points in succession on a circumference, and arcs AB, BC, and CD are equal, prove chord $AC =$ chord BD.

Prop. V. Theorem

159. (Converse of Prop. IV.) *In equal circles, or in the same circle, equal arcs are subtended by equal chords.*

Given, in equal ⓢ AMB and $A'M'B'$, arc $AB =$ arc $A'B'$; and chords AB and $A'B'$. (Fig. of Prop. IV.)

To Prove chord $AB =$ chord $A'B'$.

(The $\triangle$ ABC and $A'B'C'$ are equal by § 46.)

Prop. VI. Theorem

160. *In equal circles, or in the same circle, the greater of two arcs is subtended by the greater chord; each arc being less than a semi-circumference.*

Draw ⓢ AMB and $A'M'B'$, having same radius, and centres at C and C', respectively. Take arc $AB >$ arc $A'B'$, each arc $<$ a semi-circumference. Draw lines AB and $A'B'$. We then have:

Given, in equal ⓢ AMB and $A'M'B'$, arc $AB >$ arc $A'B'$, each arc $<$ a semi-circumference, and chords AB and $A'B'$.

To Prove chord $AB >$ chord $A'B'$.

Proof. 1. Draw radii AC, BC, $A'C'$, and $B'C'$.

2. In $\triangle$ ABC and $A'B'C'$, $AC = A'C'$, and $BC = B'C'$. (?)

3. Since, by hyp., arc $AB >$ arc $A'B'$,

$$\angle C > \angle C'. \qquad (§ 157, 2)$$

4. Then, chord $AB >$ chord $A'B'$. (§ 100)

PROP. VII. THEOREM

161. (Converse of Prop. VI.) *In equal circles, or in the same circle, the greater of two chords subtends the greater arc; each arc being less than a semi-circumference.*

Given, in equal ⓢ AMB and $A'M'B'$, chord $AB >$ chord $A'B'$, arcs AB and $A'B'$ being $<$ a semi-circumference. (Fig. of Prop. VI.)

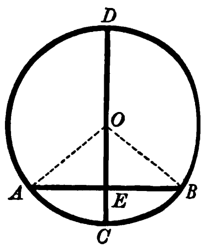
To Prove arc $AB >$ arc $A'B'$.

(We have $\angle C > \angle C'$, by § 101; then use § 157, 1.)

Ex. 4. A central angle AOB is bisected by a radius OC. Let A', C', B' be the middle points of OA, OC, and OB, respectively. Prove the length of the broken line $A'C'B'$ equal to the length of the chord AC.

PROP. VIII. THEOREM

162. *The diameter perpendicular to a chord bisects the chord and its subtended arcs.*



Draw ⊙ ABD with centre at O; draw diameter DC; through point E on OC draw chord $AB \perp DC$. We then have:

Given, in ⊙ ABD, diameter $CD \perp$ chord AB.

To Prove that CD bisects chord AB, and arcs ACB and ADB.

Proof. 1. Let O be centre of ⊙; draw radii OA and OB.

2. Since $OA = OB$, $\triangle\ OAB$ is isosceles.

3. Then, CD bisects AB, and $\angle AOB$. (§ 91)

4. Since $\angle AOC = \angle BOC$, arc $AC =$ arc BC. (§ 155)

5. Again, $\angle AOD = \angle BOD$. (§ 41, 2)

6. Then, arc $AD =$ arc BD. (?)

163. It follows from the above that *the perpendicular erected at the middle point of a chord passes through the centre of the circle, and bisects the arcs subtended by the chord.*

Prop. IX. Theorem

164. *In the same circle, or in equal circles, equal chords are equally distant from the centre.*

Draw a ⊙ with centre at O; draw equal chords AB and CD. Draw lines OE and $OF \perp AB$ and CD, respectively, and meeting these lines at E and F, respectively. We then have:

Given AB and CD equal chords of ⊙ ABC, whose centre is O, and lines OE and $OF \perp AB$ and CD, respectively.

To Prove $OE = OF$.

Proof. 1. Draw radii OA and OC.

2. In rt. $\triangle$ OAE and OCF, $OA = OC$. (?)

3. E is the middle point of AB, and F of CD. (§ 162)

4. Then, $AE = CF$;

for they are halves of equal chords AB and CD, respectively.

5. Then, $\triangle OAE = \triangle OCF$. (?)

6. Then, $OE = OF$. (?)

Ex. 5. If two equal chords, not diameters, intersect within a circle, and perpendiculars be drawn from the centre to these chords, the line joining the centre of the circle with the intersection of these chords is a bisector of the angle formed by the perpendiculars.

Ex. 6. If two circles are concentric, and a chord of the greater is a secant of the smaller, then the portions of the chord between the two circumferences are equal.

Ex. 7. The line which bisects a chord of a circle and its subtended arc will, if extended, pass through the centre of the circle.

Prop. X. Theorem

165. (Converse of Prop. IX.) *In the same circle, or in equal circles, chords equally distant from the centre are equal.*

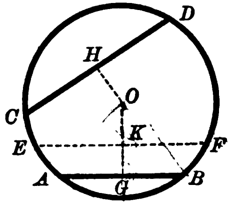
Given O the centre of ⊙ ABC, and AB and CD chords equally distant from O. (Fig. of Prop. IX.)

To Prove chord AB = chord CD.

(The rt. $\triangle$ OAE and OCF are equal (?), and $AE = CF$; E is the middle point of AB, and F of CD.)

Prop. XI. Theorem

166. *In the same circle, or in equal circles, if two chords are unequally distant from the centre, the more remote is the less.*



Draw figure in accordance with the statement. We then have:

Given O the centre of $\odot ABC$, and chord AB more remote from O than chord CD.

To Prove chord $AB <$ chord CD.

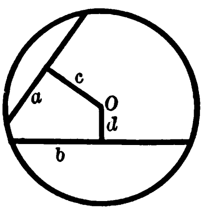
Proof. 1. Draw lines $OG \perp AB$ and $OH \perp CD$; on OG take $OK = OH$, and through K draw chord $EF \perp OK$.

2. Then, chord $EF =$ chord CD. (§ 165)
3. Now, chord $AB \parallel$ chord EF. (§ 70)
4. Then, arc AB must be $<$ arc EF.
5. Then, chord $AB <$ chord EF. (§ 160)
6. That is, chord $AB <$ chord CD.

167. By § 166, *a diameter is greater than any other chord.*

Prop. XII. Theorem

168. (Converse of Prop. XI.) *In the same circle, or in equal circles, the less of two chords is at the greater distance from the centre.*



Draw $\odot$ with centre at O; draw chords a and b, a being $< b$, their $\perp$ distances from O being c and d, respectively. We then have:

Given, in $\odot$ whose centre is O, chords a and b, respectively, at distances c and d from O, and chord $a <$ chord b.

To Prove $c > d$.

Proof. 1. If c is $< d$, chord a will be $>$ chord b. (§ 166)

2. If $c = d$, chord a will equal chord b. (§ 165)

3. Both of these results are contrary to the hyp. that chord a is $<$ chord b.

4. Then, $c > d$.

PROP. XIII. THEOREM

169. *A straight line perpendicular to a radius of a circle at its extremity is tangent to the circle.*

Draw a ⊙ with centre at O, and radius OC. Through point C draw line $AB \perp OC$. We then have:

Given line $AB \perp$ radius OC of ⊙ O at C.

To Prove AB tangent to the ⊙.

Proof. 1. Let D be any point of AB except C.

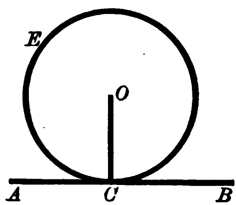
2. Draw line OD; then, $OD > OC$. (?)

3. Therefore, point D lies without the ⊙.

4. Then, every point of AB except C lies without the ⊙, and AB is tangent to the ⊙. (§ 150)

PROP. XIV. THEOREM

170. (Converse of Prop. XIII.) *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*



Draw ⊙ with centre at O. Draw line AB tangent to the ⊙ at C. Draw line OC. We then have:

Given line AB tangent to ⊙ EC at C, and radius OC.

To Prove $OC \perp AB$.

(OC is the shortest line from O to AB.)

171. It follows from § 170 that *a line perpendicular to a tangent at its point of contact passes through the centre of the circle.*

Prop. XV. Theorem

172. *Two parallels intercept equal arcs on a circumference.*

Case I. *When one line is a tangent and the other a secant.*

Draw a ⊙. Through E, any point on the circumference, draw diameter EF, and tangent $AB \perp EF$ (§ 169). Draw secant $CD \parallel AB$, intersecting the circumference at C and D. We then have:

Given AB a tangent to ⊙ CED at E, and CD a secant $\parallel AB$, intersecting the circumference at C and D.

To Prove arc $CE =$ arc DE.

Proof. 1. Draw diameter EF; then $EF \perp AB$. (§ 170)

2. Then, $EF \perp CD$. (?)

3. Then, arc $CE =$ arc DE. (§ 162)

Case II. *When both lines are secants.*

Draw a ⊙. Draw $\parallel$ secants, AB and CD, intersecting the circumference at A and B, and C and D, respectively. We then have:

Given, in ⊙ ABC, AB and CD $\parallel$ secants, intersecting the circumference at A and B, and C and D, respectively.

To Prove arc $AC =$ arc BD.

Proof. 1. Draw tangent $EF \parallel AB$, touching the ⊙ at G.

2. Then, $EF \parallel CD$. (?)

3. By Case I, arc $AG =$ arc BG, (1)

and arc $CG =$ arc DG. (2)

4. Subtracting (2) from (1),

$$\text{arc } AG - \text{arc } CG = \text{arc } BG - \text{arc } DG,$$

or

$$\text{arc } AC = \text{arc } BD.$$

Case III. *When both lines are tangents.*

In ⊙ EGF, draw ∥ tangents, AB and CD, touching ⊙ at E and F, respectively. We then have:

Given, in ⊙ EGF, AB and CD ∥ tangents, touching the ⊙ at E and F, respectively.

To Prove arc EGF = arc EHF.

(Draw secant $GH \parallel AB$, and apply Case I.)

173. It follows from § 172 that *the straight line joining the points of contact of two parallel tangents is a diameter.*

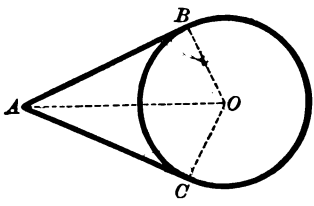
Ex. 8. The diameters of two concentric circles are 11 and 15 inches, respectively. Can a tangent to either circle be drawn through a point 5 inches from the common centre? Why?

Ex. 9. The lines joining the extremities of two unequal parallel chords form, with the chords, an isosceles trapezoid. Is the centre within or without the trapezoid?

Ex. 10. If two equal chords, AB and CD, of a circle are parallel, the points A, B, D, and C being in succession on the circumference, the chords AC and BD are equal and the figure formed is a rectangle. Can the centre be without the rectangle?

Prop. XVI. Theorem

174. *The tangents to a circle from an outside point are equal.*



Draw ⊙ with centre at O. Draw radii OB and OC, and tangents ⊥ OB and OC, respectively, intersecting at A. We then have:

Given AB and AC tangent at points B and C, respectively, to ⊙ BC whose centre is O.

To Prove $AB = AC$.

(Rt. ▵ OAB and OAC are equal by § 61.)

175. From equal $\triangle$ OAB and OAC (Fig. of Prop. XVI),

$$\angle OAB = \angle OAC \text{ and } \angle AOB = \angle AOC.$$

Then, *the line joining the centre of a circle to the point of intersection of two tangents makes equal angles with the tangents, and also with the radii drawn to the points of contact.*

Ex. 11. The line through the middle points of two parallel chords passes through the centre of the circle.

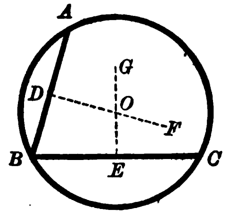
Ex. 12. If two equal circles intersect and parallel lines be drawn through the points of intersection, the portion cut from the first line by one circumference is equal to the portion cut from the second by the other circumference.

Ex. 13. The angle formed by two tangents to a circle is the supplement of the angle formed by joining the centre to the points of tangency.

Ex. 14. Can a trapezoid be constructed whose sides are 8, 10, 11, 12, respectively? Give method showing the result. Can such a trapezoid be inscribed in a circle?

Prop. XVII. Theorem

176. *Through three points, not in the same straight line, a circumference can be drawn, and but one.*



Given points A, B, and C, not in the same straight line.

To Prove that a circumference can be drawn through A, B, and C, and but one.

Proof. 1. Draw lines AB and BC, and lines DF and $EG \perp$ AB and BC, respectively, at their middle points, meeting at O.

2. Then O is equally distant from A, B, and C. (§ 136)

3. Hence, a circumference described with O as a centre and OA as a radius will pass through A, B, and C.

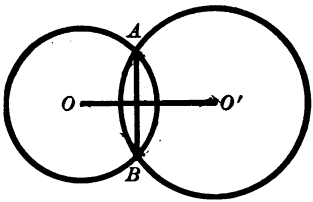
4. Again, the centre of any circumference drawn through A, B, and C must be in each of the ⊥s DF and EG. (§ 56)

5. Then as DF and EG intersect in but one point, only one circumference can be drawn through A, B, and C.

177. *Two circumferences can intersect in but two points;* for if they had three common points, they would have the same centre, and coincide throughout.

Prop. XVIII. Theorem

178. *If two circumferences intersect, the straight line joining their centres bisects their common chord at right angles.*



Draw ⓢ, with centres at O and O', whose circumferences intersect at A and B. Draw lines AB and OO'. We then have:

Given O and O' the centres of two ⓢ, whose circumferences intersect at A and B, and lines OO' and AB.

To Prove that OO' bisects AB at rt. ∠s.

(Use § 56.)

Prop. XIX. Theorem

179. *If two circles are tangent to each other, the straight line joining their centres passes through their point of contact.*

Draw two ⓢ, whose centres are at O and O', tangent to line AB at A. We then have:

Given O and O' the centres of two ⓢ, which are tangent to line AB at A.

To Prove that str. line joining O and O' passes through A.

(Draw radii OA and $O'A$; these lines are ⊥ AB, and OAO' is a str. line by § 20. Then use Ax. 6.)

Ex. 15. If two circumferences intersect, the distance between their centres is greater than the difference of their radii.

Ex. 16. If a tangent has its extremities in two parallel tangents to the same circle, the angle which is formed by the lines joining the centre of the circle to the extremities of the first tangent is a right angle.

[Draw radii to the points of tangency.]

MEASUREMENT

180. The *ratio* of one magnitude to another of the same kind is the quotient of the first divided by the second.

Thus, if a and b are quantities of the same kind, the ratio of a to b is $\frac{a}{b}$.

We *measure* a magnitude by finding its ratio to another magnitude of the same kind, called the *unit of measure.*

If the quotient can be obtained exactly as an integer or fraction, we call it the *numerical measure* of the magnitude.

181. Two magnitudes of the same kind are said to be *commensurable* when a unit of measure (called a *common measure*) is contained an *integral* number of times in each.

Thus, two lines whose lengths are $2\frac{3}{4}$ and $3\frac{4}{5}$ inches are commensurable; for the common measure $\frac{1}{20}$ inch is contained an integral number of times in each; *i.e.* 55 times in the first line, and 76 times in the second.

Two magnitudes of the same kind are said to be *incommensurable* when no magnitude of the same kind can be found which is contained an integral number of times in each.

For example, if AB and CD are two lines such that

$$\frac{AB}{CD} = \sqrt{2};$$

since $\sqrt{2}$ can only be obtained *approximately* as a decimal, no line, *however small*, can be found which is contained an integral number of times in each line, and AB and CD are incommensurable.

182. A magnitude which is incommensurable with respect to the unit has no exact numerical measure (§ 180).

Still if CD is the unit of measure, and $\frac{AB}{CD} = \sqrt{2}$, we shall call $\sqrt{2}$ the numerical measure of AB.

183. It is evident from the above that the ratio of two magnitudes of the same kind, whether commensurable or incommensurable, is equal to the ratio of their numerical measures when referred to a common unit.

THE METHOD OF LIMITS

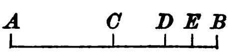
184. We define a *variable* as a quantity which, under the conditions imposed upon it, may have an indefinitely great number of *different values.*

We define a *constant* as a quantity which remains unchanged throughout the same discussion.

185. A *limit* of a variable is a constant quantity, the difference between which and the variable may be made less than any assigned quantity, however small.

In other words, a limit of a variable is a fixed quantity to which the variable approaches indefinitely near.

186. Suppose, for example, that a point moves from A towards B under the condition that it shall move, during successive equal intervals of time, first from A to C, halfway between A and B; then to D, halfway between C and B; then to E, halfway between D and B; and so on indefinitely.



In this case, the distance between the moving point and B can be made less than any assigned distance, however small.

Then, the distance from A to the moving point is a variable which approaches the constant distance AB as a limit.

Again, the distance from the moving point to B is a variable which approaches the limit 0.

As another illustration, consider the series

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \cdots,$$

where each term after the first is one-half the preceding.

In this case, by taking terms enough, the last term may be made less than any assigned number, however small.

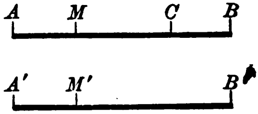
Then, the last term of the series is a variable which approaches the limit 0 when the number of terms is indefinitely increased.

Again, the sum of the first two terms is $1\frac{1}{2}$;
the sum of the first three terms is $1\frac{3}{4}$;
the sum of the first four terms is $1\frac{7}{8}$; etc.

In this case, by taking terms enough, the sum of the terms may be made to differ from 2 by less than any assigned number, however small.

Then, the sum of the terms of the series is a variable which approaches the limit 2 when the number of terms is indefinitely increased.

187. The Theorem of Limits. *If two variables are always equal, and each approaches a limit, the limits are equal.*



Given AM and $A'M'$ two variables, which are always equal, and approach the limits AB and $A'B'$, respectively.

To Prove $AB = A'B'$.

Proof. 1. If possible, suppose $AB > A'B'$.

2. Take, on AB, AC equal to $A'B'$.

3. Then, variable AM may assume values $> AC$, while variable $A'M'$ is restricted to values $< AC$.

4. This is contrary to the hypothesis that the variables are always equal; then, AB cannot be $> A'B'$.

5. In like manner, we may prove that AB cannot be $< A'B'$.

6. Since AB cannot be $>$, nor $<$ $A'B'$, we have $AB = A'B'$.

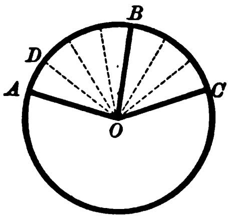
Note. In the above demonstration, we have supposed AM and $A'M'$ to be *increasing* variables; the theorem may be proved in a similar manner if AM and $A'M'$ are *decreasing* variables.

MEASUREMENT OF ANGLES

Prop. XX. Theorem

188. *In the same circle, or in equal circles, two central angles are in the same ratio as their intercepted arcs.*

Case I. *When the arcs are commensurable* (§ 181).



Draw figure in accordance with the statement. We then have:

Given, in ⊙ ABC, AOB and BOC central ∡ intercepting commensurable arcs AB and BC, respectively.

To Prove $$\frac{\angle AOB}{\angle BOC}=\frac{\text{arc } AB}{\text{arc } BC}.$$

Proof. 1. By hyp., arcs AB and BC are commensurable; let arc AD be a common measure of arcs AB and BC, and suppose it to be contained 4 times in arc AB, and 3 times in arc BC.

2. Then, $$\frac{\text{arc } AB}{\text{arc } BC}=\frac{4}{3}. \quad (1)$$

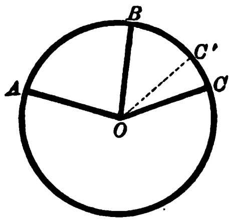
3. Drawing radii to the several points of division of arc AC, $\angle AOB$ will be divided into 4 ∡, and $\angle BOC$ into 3 ∡, all of which ∡ are equal. (§ 155)

4. Then, $$\frac{\angle AOB}{\angle BOC}=\frac{4}{3}. \quad (2)$$

5. From (1) and (2), $$\frac{\angle AOB}{\angle BOC}=\frac{\text{arc } AB}{\text{arc } BC}. \quad (?)$$

Note. The theorem may be proved in a similar manner, whatever the number of subdivisions of arcs AB and BC.

Case II. *When the arcs are incommensurable* (§ 181).



Draw figure in accordance with the statement. We then have:

Given, in ⊙ ABC, AOB and BOC central ∠s intercepting incommensurable arcs AB and BC, respectively.

To Prove $$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}.$$

Proof. 1. Let arc AB be divided into any number of equal arcs, and let one of these arcs be applied to arc BC as a unit of measure.

2. Since arcs AB and BC are incommensurable, a certain number of equal arcs will extend from B to C', leaving a remainder $C'C$ less than one of the equal arcs.

3. Draw radius OC'; since arcs AB and BC' are commensurable,

$$\frac{\angle AOB}{\angle BOC'} = \frac{\text{arc } AB}{\text{arc } BC'}. \qquad \text{(Case I)}$$

4. Now let the number of subdivisions of arc AB be indefinitely increased.

5. Then the unit of measure will be indefinitely diminished; and the remainder $C'C$, being always less than the unit, will approach the limit 0.

6. Then $\angle BOC'$ will approach the limit $\angle BOC$,

and arc BC' will approach the limit arc BC.

7. Hence, $\dfrac{\angle AOB}{\angle BOC'}$ will approach the limit $\dfrac{\angle AOB}{\angle BOC}$,

and $\dfrac{\text{arc } AB}{\text{arc } BC'}$ will approach the limit $\dfrac{\text{arc } AB}{\text{arc } BC}$.

8. Now, $\frac{\angle AOB}{\angle BOC'}$ and $\frac{\text{arc } AB}{\text{arc } BC'}$ are two variables which are always equal; and they approach the limits $\frac{\angle AOB}{\angle BOC}$ and $\frac{\text{arc } AB}{\text{arc } BC}$, respectively.

9. By the Theorem of Limits, these limits are equal. (§ 187)

10. Then,
$$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}.$$

189. The usual unit of measure for arcs is the *degree*, which is the ninetieth part of a quadrant (§ 154).

The degree of arc is divided into sixty equal parts, called *minutes*, and the minute into sixty equal parts, called *seconds*.

If the sum of two arcs is a quadrant, or 90°, one is called the *complement* of the other; if their sum is a semi-circumference, or 180°, one is called the *supplement* of the other.

190. By § 155, equal central $\angle$s, in the same ⊙, intercept equal arcs on the circumference.

Hence, if the angular magnitude about the centre of a ⊙ be divided into four equal $\angle$s, each $\angle$ will intercept an arc equal to one-fourth of the circumference.

That is, *a right central angle intercepts a quadrant on the circumference.* (§ 19)

191. By § 188, a central $\angle$ of n degrees bears the same ratio to a rt. central $\angle$ that its intercepted arc bears to a quadrant.

But a central $\angle$ of n degrees is $\frac{n}{90}$ of a rt. central $\angle$.

Hence, its intercepted arc is $\frac{n}{90}$ of a quadrant, or an arc of n degrees.

The above principle is usually expressed as follows:

A central angle is measured by its intercepted arc.

The above statement signifies simply that the number of angular degrees in a central angle is equal to the number of degrees of arc in its intercepted arc.

PROP. XXI. THEOREM

192. *An inscribed angle is measured by one-half its intercepted arc.*

Case I. *When one side of the angle is a diameter.*

Draw ⊙ ABC with centre at O. Draw diameter AC and chord AB. We then have:

Given AC a diameter, and AB a chord, of ⊙ ABC.

To Prove that $\angle A$ is measured by $\frac{1}{2}$ arc BC.

Proof. 1. Draw radius OB; denote $\angle BOC$ by a.

2. We have, $OA = OB$. (?)

3. Then, $\angle B = \angle A$. (§ 50)

4. Since a is an ext. $\angle$ of $\triangle OAB$, $\angle a = \angle A + \angle B$. (§ 85)

5. Then, $\angle a = 2\angle A$, or $\angle A = \frac{1}{2}\angle a$.

6. But, $\angle a$ is measured by arc BC. (§ 191)

7. Whence, $\angle A$ is measured by $\frac{1}{2}$ arc BC.

Case II. *When the centre is within the angle.*

Draw ⊙ ABC; draw an inscribed $\angle BAC$ with the centre within the $\angle$. We then have:

Given AB and AC chords of ⊙ ABC, and the centre of the ⊙ within $\angle BAC$.

To Prove that $\angle BAC$ is measured by $\frac{1}{2}$ arc BC.

Proof. 1. Draw diameter AD; denote $\angle BAD$ by a and $\angle BAC$ by b.

2. By Case I, $\angle a$ is measured by $\frac{1}{2}$ arc BD,

and $\angle b$ is measured by $\frac{1}{2}$ arc CD.

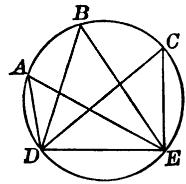
3. Then, $\angle a + \angle b$ is measured by $\frac{1}{2}$ arc $BD + \frac{1}{2}$ arc CD.

4. Then, $\angle BAC$ is measured by $\frac{1}{2}$ arc BC.

Case III. *When the centre is without the angle.*

Draw ⊙ ABC. Inscribe $\angle BAC$, the centre lying without the $\angle$. Draw diameter AD.

(The proof is left to the pupil.)

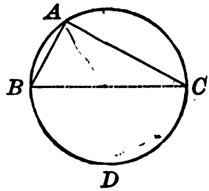


193. *Angles inscribed in the same segment are equal.*

For, if A, B, and C are $\angle$s inscribed in segment AED of $\odot ABC$,

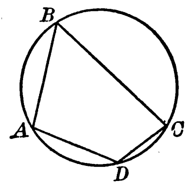
$$\angle A = \angle B = \angle C;$$

for each $\angle$ is measured by $\frac{1}{2}$ arc DE (§ 192).



194. *An angle inscribed in a semicircle is a right angle.*

For if BC be a diameter, and AB and AC chords, of $\odot ABD$, $\angle A$ is measured by $\frac{1}{2}$ an arc of 180°, or 90° (§ 192).



195. *The opposite angles of an inscribed quadrilateral are supplementary.*

For their sum is measured by $\frac{1}{2}$ of 360°, or 180°. (?)

Ex. 17. A quadrilateral $ABCD$ is inscribed in a circle. Arc AB is 84°, arc CD is 96°, and $\angle C$ is 80°. Find remaining arcs and angles.

Ex. 18. A quadrilateral $ABCD$ is inscribed in a circle. $\angle B$ is 104°, $\angle C$ is 92°, and arc AB 70°. Find remaining angles of the quadrilateral and the arcs subtended by its sides.

Ex. 19. A regular polygon has seven sides. Find the number of degrees in each angle.

Ex. 20. The median drawn to the hypotenuse of a right triangle divides it into two isosceles triangles.

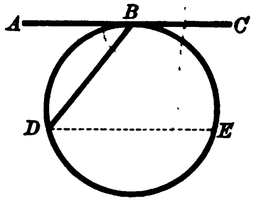
Ex. 21. If angles be inscribed in two segments whose sum is equal to the whole circle, they are supplements and each angle is a supplement of one-half the central angle which its arc subtends.

Ex. 22. A triangle is inscribed in a circle; one of its angles is 74°, and one side of this angle subtends an arc of 42°; find the remaining angles of the triangle and the subtended arcs.

Ex. 23. One of the equal sides of an isosceles triangle ABC is 30 inches, the vertical angle A is 120°. A circle described on one of the equal sides as a diameter intersects BC at D. Find BD in terms of BC.

Prop. XXII. Theorem

196. *The angle formed by a tangent and a chord is measured by one-half its intercepted arc.*



Draw a ⊙. Draw tangent AC, touching the ⊙ at B, and chord BD making an acute $\angle$ with AB. We then have:

Given AC tangent to ⊙ BDE at B, and BD a chord.

To Prove that $\angle ABD$ is measured by $\frac{1}{2}$ arc BD.

Proof. 1. Draw chord $DE \parallel AC$; then, $\angle ABD = \angle D$. (§ 75)

2. Now $\angle D$ is measured by $\frac{1}{2}$ arc BE. (§ 192)

3. Also, arc BE = arc BD. (§ 172)

4. Then, $\angle ABD$ is measured by $\frac{1}{2}$ arc BD.

Since $\angle CBD$ is sup. to $\angle ABD$, it is measured by $\frac{1}{2}$ (a circumference − arc BD), or $\frac{1}{2}$ arc BED. (§ 192)

Prop. XXIII. Theorem

197. *The angle formed by two chords, intersecting within the circumference, is measured by one-half the sum of its intercepted arc, and the arc intercepted by its vertical angle.*

Draw ⊙ ABC. Draw chords AB and CD intersecting within the circumference at E; denote $\angle AEC$ by a. We then have:

Given, in ⊙ ABC, chords AB and CD intersecting within the circumference at E.

To Prove that $\angle a$ is measured by $\frac{1}{2}$ (arc AC + arc BD).

Proof. 1. Draw chord BC; since a is an ext. $\angle$ of $\triangle BCE$,

$$\angle a = \angle B + \angle C. \quad (?)$$

2. But, $\angle B$ is measured by $\frac{1}{2}$ arc AC,

and $\angle C$ is measured by $\frac{1}{2}$ arc BD. (?)

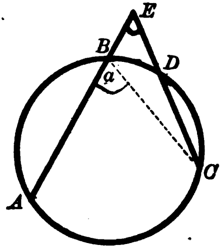
3. Substituting in result of (1),

$$\angle a \text{ is measured by } \tfrac{1}{2} (\text{arc } AC + \text{arc } BD).$$

Ex. 24. In an inscribed quadrilateral $ABCD$, $\angle A = 75°$, $\angle B = 40°$, arc $BC = 95°$; find the angle formed by the diagonals of the quadrilateral.

Prop. XXIV. Theorem

198. *The angle formed by two secants, intersecting without the circumference, is measured by one-half the difference of the intercepted arcs.*



Draw $\odot ABC$. Draw secants AE and CE intersecting without the circumference at E, and intersecting the circumference at A and B, and C and D, respectively. We then have:

Given, in $\odot ABC$, secants AE and CE intersecting without the circumference at E, and intersecting the circumference at A and B, and C and D, respectively.

To Prove that $\angle E$ is measured by $\frac{1}{2}$ (arc AC − arc BD).

Proof. 1. Draw chord BC; denote $\angle ABC$ by a.

2. Since a is an ext. $\angle$ of $\triangle BCE$, $\angle a = \angle E + \angle C$. (?)

3. Then, $\angle E = \angle a - \angle C$.

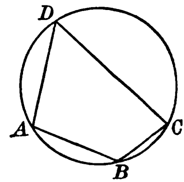
4. Now $\angle a$ is measured by $\frac{1}{2}$ arc AC,

and $\angle C$ is measured by $\frac{1}{2}$ arc BD. (?)

5. Then, $\angle E$ is measured by $\frac{1}{2}$ (arc AC − arc BD).

199. *If the opposite angles of a quadrilateral are supplementary, the quadrilateral can be inscribed in a circle.*

(Converse of § 195.)



Suppose, in quadrilateral $ABCD$, $\angle A$ sup. to $\angle C$, and $\angle B$ to $\angle D$; and a circumference drawn through A, B, and C. (§ 176)

If $\angle D$ is sup. to $\angle B$, it is measured by $\frac{1}{2}$ arc ABC. (§ 192)

Then, D must lie on the circumference; for if it were within the $\odot$, $\angle D$ would be measured by $\frac{1}{2}$ an arc $> ABC$; and if it were without the $\odot$, $\angle D$ would be measured by $\frac{1}{2}$ an arc $< ABC$. (§§ 197, 198)

Ex. 25. Two equal chords bisect in a circle. What are the angles formed?

Ex. 26. Two chords intersect within a circle, and form an angle of 75°; one of the intercepted arcs is 60°: find the other.

Ex. 27. Two chords intersect within a circle; one of the angles formed is $A°$, and one of the intercepted arcs is $B°$: find the other.

Prop. XXV. Theorem

200. *The angle formed by a secant and a tangent, or two tangents, is measured by one-half the difference of the intercepted arcs.*

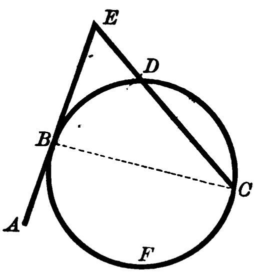


Fig. 1.

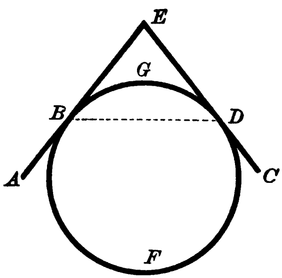


Fig. 2.

Draw figures in accordance with the statement, AE tangent to $\odot$ BDF at B, and CE a chord in Fig. 1, a secant in Fig. 2. We then have:

1. **Given** AE a tangent to $\odot BDC$ at B, and EC a secant intersecting the circumference at C and D. (Fig. 1.)

To Prove that $\angle E$ is measured by $\frac{1}{2}$ (arc BFC – arc BD).

(We have $\angle E = \angle ABC - \angle C$.)

2. (In Fig. 2, $\angle E = \angle ABD - \angle BDE$; then use § 197.)

Ex. 28. In Ex. 27 find the point of intersection of these chords when A and B are equal.

Ex. 29. Two chords intersecting in a circle intercept opposite arcs on the circumference 58° and 14°, respectively. Find the angle formed by the chords.

Ex. 30. Two unequal chords intersect at right angles. Find the sum of the opposite intercepted arcs. Will this sum remain constant if the lengths of the chords vary?

Ex. 31. A chord subtends an arc of 144°. Through one extremity of the chord a tangent is drawn; find the angles which it forms with the chord.

Ex. 32. The angle formed by a tangent and a secant is 94°, and one of the intercepted arcs is 210°; find the other.

Ex. 33. Find the angle formed by two secants intersecting without a circle, if the intercepted arcs are 36° and 144°, respectively; also if the angle is 75° and one of the intercepted arcs is 44°, find the other.

Ex. 34. An inscribed angle is 32°. How many degrees in the arc in which it is inscribed?

Ex. 35. In how large a circle can a triangle whose sides are 12, 13, and 5 be inscribed? What kind of a triangle is it?

Ex. 36. The angle formed by two tangents is 76°; find the intercepted arcs; also for 92°.

Ex. 37. An inscribed quadrilateral subtends arcs of 70°, 80°, 100°, and 110°, respectively. Find the angles of the quadrilateral, and also the angles of the quadrilateral formed by the intersections of the tangents through the vertices of the quadrilateral.

Ex. 38. The sum of the degrees in the angle formed by two tangents and the smaller of the intercepted arcs is 180°.

Ex. 39. Choose any two of the following statements for the hypothesis and any one of the remaining three for the conclusion, and prove each of the ten theorems thus formed, not already proved in §§ 162, 163:

A straight line that

1. Passes through the centre of a circle;
2. Bisects the chord;
3. Is perpendicular to the chord;
4. Bisects the smaller arc;
5. Bisects the greater arc.

Ex. 40. Give a general rule for the measurement of angles formed by two lines intersecting

(*a*) within a circumference,
(*b*) on a circumference,
(*c*) without a circumference.

Ex. 41. What figure is formed by the bisectors of the angles of a parallelogram? Give proof.

Ex. 42. Two equal chords intersect either within a circumference, on the circumference, or without the circumference. Prove that in either case the bisector of one of the angles formed by the chords passes through the centre of the circle.

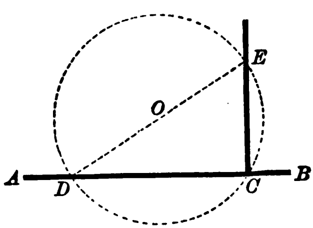
Ex. 43. Given a chord CD and A and B, two points on the circumference, such that line $AC = AD$ and line $BC = BD$, prove that line AB is a diameter.

CONSTRUCTIONS

Prop. XXVI. Problem

201. *At a given point in a straight line to erect a perpendicular to that line.*

In § 24, we gave a method for drawing a perpendicular to a straight line at a given point; the following method is advantageous when the point is near the end of the line.



Given C any point in line AB.

Required to draw a line $\perp$ AB at C.

Construction. 1. With any point O without line AB as a centre, and distance OC as a radius, describe a circumference intersecting AB at C and D.

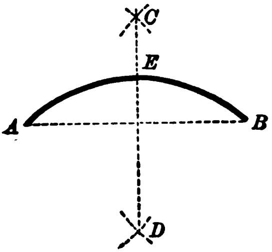
2. Draw diameter DE, and line CE.

3. Then, CE is $\perp$ AB at C.

Proof. 4. $\angle$ DCE, being inscribed in a semicircle, is a rt. $\angle$. (§ 194)

PROP. XXVII. PROBLEM

202. *To bisect a given arc.*



Given arc AB.

Required to bisect arc AB.

Construction. 1. With A and B as centres, and with equal radii, describe arcs intersecting at C and D.

2. Draw line CD intersecting arc AB at E.

3. Then E is the middle point of arc AB.

Proof. 4. Draw chord AB.

5. Then, CD is $\perp$ chord AB at its middle point. (?)

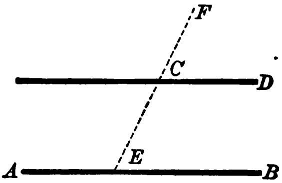
6. Whence, CD bisects arc AB. (§ 163)

Ex. 44. The hypotenuse of an isosceles right triangle is 18. Construct the triangle.

Ex. 45. Given two angles of a triangle, to construct the third. (Compare § 84.)

PROP. XXVIII. PROBLEM

203. *Through a given point without a given straight line, to draw a parallel to the line.*



Given C any point without line AB.

Required to draw through C a line $\parallel AB$.

Construction. Through C draw any line EF, meeting AB at E, and construct $\angle FCD = \angle CEB$ (§ 28); then, $CD \parallel AB$.

(The proof is left to the pupil.)

PROP. XXIX. PROBLEM

204. *Given two sides of a triangle, and the angle opposite to one of them, to construct the triangle.*

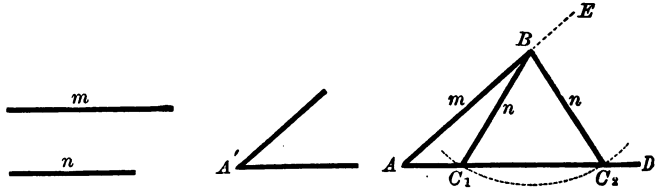
Given m and n sides of a $\triangle$, and A' the $\angle$ opposite to n.

Required to construct the $\triangle$.

Construction. 1. Construct $\angle DAE = \angle A'$ (§ 28); on AE take $AB = m$.

2. With B as a centre and n as a radius, describe an arc.

Case I. *When A' is acute, and $m > n$.*



There may be three cases:

I. The arc may intersect AD in two points.

3. Let C_1 and C_2 be the points in which the arc intersects AD, and draw lines BC_1 and BC_2.

4. Then, either ABC_1 or ABC_2 is the required $\triangle$.

This is called the *ambiguous case.*

II. The arc may be tangent to AD.

In this case there is but one $\triangle$.

5. Since a tangent to a $\odot$ is $\perp$ the radius drawn to the point of contact (§ 170), the $\triangle$ is a *right* $\triangle$.

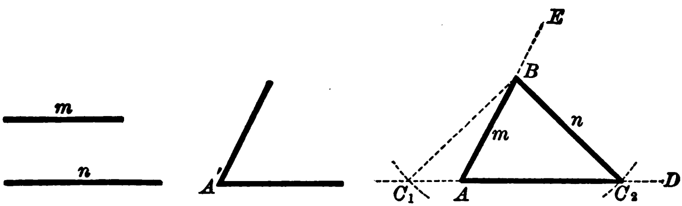
III. The arc may not intersect AD at all.

In this case the problem is impossible.

Case II. *When A' is acute, and $m = n$.*

6. In this case, the arc intersects AD in two points, of which A is one; there is but one $\triangle$, an isosceles $\triangle$.

Case III. *When A' is acute, and $m < n$.*



7. In this case, the arc intersects AD in two points.

8. Let C_1 and C_2 be the points in which the arc intersects AD, and draw lines BC_1 and BC_2.

9. Now $\triangle ABC_1$ does not satisfy the conditions of the problem, since it does not contain $\angle A'$.

10. Then there is but one $\triangle$; $\triangle ABC_2$.

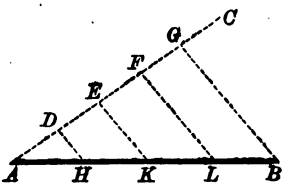
Case IV. *When A' is right or obtuse, and $m < n$.*

11. In each of these cases, the arc intersects AD in two points on opposite sides of A, and there is but one $\triangle$.

The pupil should construct the triangle corresponding to each case of § 204.

PROP. XXX. PROBLEM

205. *To divide a given straight line into any number of equal parts.*



Given line AB.

Required to divide AB into four equal parts.

Construction. 1. On the indefinite line AC, take any convenient length AD.

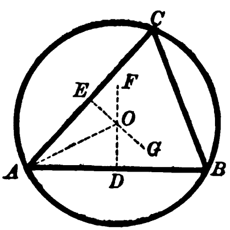
2. On DC take $DE = AD$; on EC take $EF = AD$; on FC take $FG = AD$; and draw line BG.

3. Draw lines DH, EK, and $FL \parallel BG$, meeting AB at H, K, and L, respectively.

4. Then, $\quad AH = HK = KL = LB.$ (§ 128)

PROP. XXXI. PROBLEM

206. *To circumscribe a circle about a given triangle.*



Given $\triangle ABC$.

Required to circumscribe a $\odot$ about $\triangle ABC$.

Construction. 1. Draw lines DF and $EG \perp AB$ and AC, respectively, at their middle points, intersecting at O. (§ 26

2. With O as a centre, and OA as a radius, describe a $\odot$.
3. The circumference will pass through A, B, and C.

(The proof is left to the pupil; see § 136.)

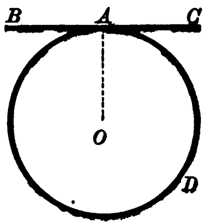
Note. The above construction serves to describe a circumference through three given points not in the same straight line, or to find the centre of a given circumference or arc.

Ex. 46. Through any given point within a circumference to draw a chord of given length.

Ex. 47. To inscribe a circle in a given triangle. (Compare § 134.)

PROP. XXXII. PROBLEM

207. *To draw a tangent to a circle through a given point on the circumference.*



Given A any point on the circumference of $\odot AD$.

Required to draw through A a tangent to $\odot AD$.

Construction. 1. Draw radius OA.

2. Through A draw line $BC \perp OA$. (§ 24)
3. Then, BC will be tangent to $\odot AD$. (?)

Ex. 48. Given two sides and the included angle of a parallelogram, to construct the parallelogram.

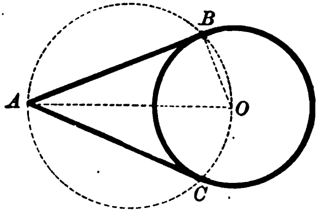
Ex. 49. Circumscribe a parallelogram about a circle and find relations between the lengths of the adjacent sides.

Ex. 50. To draw a tangent to a given circle which shall be perpendicular to a given line.

Ex. 51. To draw the shortest chord through a given point within a given circumference.

PROP. XXXIII. PROBLEM

208. *To draw a tangent to a circle through a given point without the circle.*



Given A any point without $\odot BC$.

Required to draw through A a tangent to $\odot BC$.

Construction. 1. Let O be the centre of $\odot BC$.

2. Draw line OA, and with OA as a diameter, describe a circumference, cutting the given circumference at B and C.

3. Draw lines AB and AC.

4. Then, AB and AC are tangents to $\odot BC$.

Proof. 5. Draw line OB.

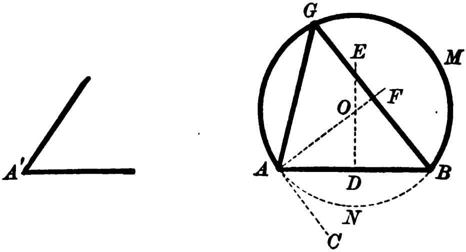
6. Since $\angle ABO$ is inscribed in a semicircle, it is a rt. $\angle$. (?)

7. Then, AB is tangent to $\odot BC$. (?)

In like manner, we may prove AC tangent to $\odot BC$.

PROP. XXXIV. PROBLEM

209. *Upon a given straight line, to describe a segment which shall contain a given angle.*



Given line AB, and $\angle A'$.

Required to describe upon AB a segment such that every $\angle$ inscribed in the segment shall equal $\angle A'$.

Construction. 1. Construct $\angle BAC = \angle A'$. (§ 28)

2. Draw line $DE \perp AB$ at its middle point. (§ 26)

3. Draw line $AF \perp AC$, intersecting DE at O.

4. With O as centre and OA as radius, describe $\odot AMBN$.

5. Then, AMB will be the required segment.

Proof. 6. If AGB be any $\angle$ inscribed in segment AMB, it is measured by $\frac{1}{2}$ arc ANB. (?)

7. But, by cons., $AC \perp OA$; and AC is tangent to $\odot$ AMB. (?)

8. Therefore, $\angle BAC$ is measured by $\frac{1}{2}$ arc ANB. (§ 196)

9. Then, $\angle AGB = \angle BAC = \angle A'$. (?)

10. Hence, every $\angle$ inscribed in segment AMB equals $\angle A'$. (§ 193)

Ex. 52. From a given point without a circle to draw two equal secants terminating in the circumference. Can more than two be drawn?

Ex. 53. Trisect a right angle.

(In general, an angle cannot be trisected by methods of Euclidean geometry.)

Ex. 54. In a circle whose radius is 12, draw a chord whose length is 8, parallel to a given line.

Ex. 55. One of the equal sides of an isosceles triangle is 8, and one of the equal angles 30°. Construct the triangle.

Ex. 56. The distance between two parallel lines is a, and a line is drawn intersecting these two parallels. Construct a circle which shall be tangent to the three lines. How many such circles are possible?

Ex. 57. The sides of a triangle are a, b, and c, respectively. The angle opposite b is twice, and the angle opposite c three times, the angle opposite a. Construct the triangle.

Ex. 58. The base of an isosceles triangle is 10, and the perimeter 18. Construct the triangle.

Ex. 59. In a rhombus $ABCD$, $\angle A$ is twice $\angle B$. The shorter diagonal is 9. Construct the rhombus.

Ex. 60. Two angles of an inscribed quadrilateral are 90° and 60°, respectively. Find the remaining angles and inscribe the quadrilateral in a given circle. Can more than one such quadrilateral be inscribed?

Ex. 61. One side of a triangle is three-fourths another, and three-fifths the third side. The perimeter of the triangle is 48. Construct the triangle.

Ex. 62. A given point lies within, without, or on the circumference of, a circle. With the given point as centre, describe a circumference passing through the extremities of a given diameter of the circle.

Ex. 63. We define the angle between two intersecting curves as the angle formed by tangents to the curves at their point of intersection. With a point outside a given circle as a centre, describe an arc which shall intersect the given circumference at right angles.

Book III

PROPORTION.—SIMILAR POLYGONS

DEFINITIONS

210. A *Proportion* is a statement that two ratios are equal.

211. The statement that the ratio of a to b is equal to the ratio of c to d is written

$$\frac{a}{b}=\frac{c}{d}.$$

212. In the proportion $\frac{a}{b}=\frac{c}{d}$, we call a the *first* term, b the *second* term, c the *third* term, and d the *fourth* term.

213. We call the first and fourth terms the *extremes*, and the second and third terms the *means*.

We call the first and third terms the *antecedents*, and the second and fourth terms the *consequents*.

Thus, in the above proportion a and d are the extremes, b and c the means, a and c the antecedents, and b and d the consequents.

214. If the means of a proportion are equal, either mean is called a *mean proportional* between the first and last terms, and the last term a *third proportional* to the first and second terms.

Thus, in the proportion $\frac{a}{b}=\frac{b}{c}$, b is a mean proportional between a and c, and c a third proportional to a and b.

215. In the proportion $\frac{a}{b}=\frac{c}{d}$, d is called a *fourth proportional* to a, b, and c.

Prop. I. Theorem

216. *In any proportion, the product of the extremes is equal to the product of the means.*

Given the proportion $\frac{a}{b}=\frac{c}{d}.$ (1)

To Prove $ad = bc.$

Proof. Multiplying both members of equation (1) by bd,

$$ad = bc.$$

217. Applying the above theorem to the proportion

$$\frac{a}{b}=\frac{b}{c}, \text{ we have } b^2 = ac, \text{ or } b = \sqrt{ac}.$$

That is, *the mean proportional between two numbers is equal to the square root of their product.*

Prop. II. Theorem

218. (Converse of Prop. I.) *If the product of two numbers is equal to the product of two others, one pair may be made the extremes, and the other pair the means, of a proportion.*

Given $ad = bc.$ (1)

To Prove $\frac{a}{b}=\frac{c}{d}.$

Proof. Dividing both members of (1) by bd,

$$\frac{ad}{bd}=\frac{bc}{bd}, \text{ or } \frac{a}{b}=\frac{c}{d}.$$

In like manner, we may prove $\frac{a}{c}=\frac{b}{d}$; $\frac{b}{a}=\frac{d}{c}$; etc.

Prop. III. Theorem

219. *In any proportion, the terms are in proportion by* Alternation; *that is, the first term is to the third as the second term is to the fourth.*

Given the proportion $\frac{a}{b}=\frac{c}{d}.$ (1)

To Prove $\frac{a}{c}=\frac{b}{d}.$

Proof. From (1), $ad=bc.$ (§ 216)

Then, $\frac{a}{c}=\frac{b}{d}.$ (§ 218)

PROP. IV. THEOREM

220. *In any proportion, the terms are in proportion by* INVERSION; *that is, the second term is to the first as the fourth term is to the third.*

Given the proportion $\frac{a}{b}=\frac{c}{d}.$ (1)

To Prove $\frac{b}{a}=\frac{d}{c}.$

Proof. From (1), $ad=bc.$ (?)

Then, $\frac{b}{a}=\frac{d}{c}.$ (?)

PROP. V. THEOREM

221. *In any proportion, the terms are in proportion by* COMPOSITION; *that is, the sum of the first two terms is to the first term as the sum of the last two terms is to the third term.*

Given the proportion $\frac{a}{b}=\frac{c}{d}.$ (1)

To Prove $\frac{a+b}{a}=\frac{c+d}{c}.$

Proof. From (1), $ad=bc.$ (?)

Adding both members of this equation to ac,

$$ac+ad=ac+bc, \text{ or } a(c+d)=c(a+b).$$

Then, $\frac{a+b}{a}=\frac{c+d}{c}$ (§ 218). (2)

In like manner, we may prove $\frac{a+b}{b}=\frac{c+d}{d}.$

PROP. VI. THEOREM

222. *In any proportion, the terms are in proportion by* DIVISION; *that is, the difference of the first two terms is to the first term as the difference of the last two terms is to the third term.*

Given the proportion $\frac{a}{b}=\frac{c}{d}$. (1)

To Prove $\frac{a-b}{a}=\frac{c-d}{c}$.

Proof. From (1), $ad=bc$. (?)

Subtracting both members from ac,

$$ac-ad=ac-bc, \text{ or } a(c-d)=c(a-b).$$

Then, $\frac{a-b}{a}=\frac{c-d}{c}$ (§ 218). (2)

In like manner, we prove $\frac{a-b}{b}=\frac{c-d}{d}$.

PROP. VII. THEOREM

223. *In any proportion, the terms are in proportion by* COMPOSITION AND DIVISION; *that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

Given the proportion $\frac{a}{b}=\frac{c}{d}$.

To Prove $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

Proof. Dividing equation (2), § 221, by equation (2), § 222,

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

Ex. 1. What is the ratio of 50 cents to \$$1\frac{1}{2}$? of $\frac{1}{2}$ hour to 1 day? of 150 rods to 1 mile?

Ex. 2. If x is a fourth proportional to a, b, c, find x.

Ex. 3. Apply composition and division to the following:

$$\frac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}=\frac{2a+b}{2a-b}.$$

PROP. VIII. THEOREM

224. *In a series of equal ratios, the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

Given $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}. \qquad (1)$$

To Prove $$\frac{a+c+e}{b+d+f}=\frac{a}{b}.$$

Proof. We have $ba = ab.$

And from (1), $bc = ad,$

and $be = af.$ (?)

Adding, $ba + bc + be = ab + ad + af.$

Or, $b(a + c + e) = a(b + d + f).$

Then, $$\frac{a+c+e}{b+d+f}=\frac{a}{b}. \qquad (?)$$

225. The ratio of two magnitudes of the same kind is equal to the ratio of their numerical measures when referred to a common unit (§ 183).

Then, in every proportion involving the ratio of magnitudes of the same kind, we shall regard the ratio of the magnitudes as replaced by *the ratio of their numerical measures when referred to a common unit.*

Thus, if AB and CD are lines, and

$$\frac{AB}{CD}=\frac{m}{n},$$

where m and n are numbers, this means simply that the *ratio of the numerical measures of AB and CD, when referred to a common unit, is equal to the ratio of m to n.*

Ex. 4. Find a third proportional to $\frac{14}{9}$ and $\frac{7}{12}$.

Ex. 5. Find a mean proportional between $\frac{25}{14}$ and $\frac{224}{9}$.

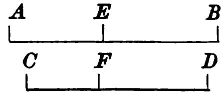
Ex. 6. If the ratio of the hypotenuse of a right triangle to one of the legs is 2, the acute angles are 30° and 60°.

Ex. 7. Each of two angles of a triangle is 45°. Find the ratio of the sides opposite these angles.

Ex. 8. Two angles of a triangle are 30° and 60°, respectively. Find the ratio of the sides opposite these angles. Is the ratio commensurable?

PROPORTIONAL LINES

226. Def. Two straight lines are said to be *divided proportionally* when their corresponding segments are in the same ratio as the lines themselves.



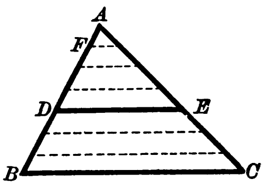
Thus, lines AB and CD are divided proportionally if

$$\frac{AE}{CF}=\frac{BE}{DF}=\frac{AB}{CD}.$$

Prop. IX. Theorem

227. *A parallel to one side of a triangle divides the other two sides proportionally.*

Case I. *When the segments of each side are commensurable.*



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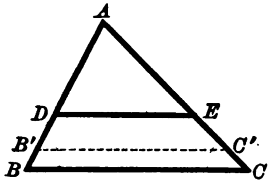
Draw $\triangle ABC$. Take D, a point in AB, such that AD and DB are commensurable; draw $DE \parallel BC$, meeting AC at E. We then have:

Given, in $\triangle ABC$, segments AD and BD of side AB commensurable, and line $DE \parallel BC$, meeting AC at E.

To Prove $$\frac{AD}{BD}=\frac{AE}{CE}.$$

Proof. 1. Let AF be contained 4 times in AD, 3 times in BD.
2. Find ratio of AD to BD.
3. Through points of division of AB draw $\parallel$s to BC.
4. Find ratio of AE to CE. (§ 128)
5. From steps (2) and (4), $\dfrac{AD}{BD}=\dfrac{AE}{CE}$. (?)

Case II. *When the segments of each side are incommensurable.*



Draw figure in accordance with the statement of the proposition. We then have:

Given, in $\triangle ABC$, segments AD and BD of side AB incommensurable, and line $DE \parallel BC$, meeting AC at E.

To Prove $$\frac{AD}{BD} = \frac{AE}{CE}.$$

Proof. 1. Let AD be divided into any number of equal parts, and let one of these parts be applied to BD as a unit of measure.

2. Since AD and BD are incommensurable, a certain number of the equal parts will extend from D to B', leaving a remainder $BB' <$ one of the equal parts.

3. Draw line $B'C' \parallel BC$, meeting AC at C'.

4. Then, AD and $B'D$ are commensurable, and

$$\frac{AD}{B'D} = \frac{AE}{C'E}. \qquad \text{(Case I)}$$

5. Now, if the number of subdivisions of AD be indefinitely increased, the unit of measure will be indefinitely diminished, and the remainder BB' will approach the limit 0.

6. Then, $\frac{AD}{B'D}$ will approach the limit $\frac{AD}{BD}$,

and $\frac{AE}{C'E}$ will approach the limit $\frac{AE}{CE}$.

7. By the Theorem of Limits, these limits are equal. (?)

8. Then, $$\frac{AD}{BD} = \frac{AE}{CE}.$$

The above result simply means that the ratio of the numerical measures of lines AD and BD, when referred to a common unit, is equal to ratio of the numerical measures of lines AE and CE, when referred to a common unit.

A similar meaning is attached to every proportion involving the ratio of two geometrical magnitudes of the same kind. (Compare § 225.)

228. Applying the theorem of § 221 to the proportion of § 227, we have

$$\frac{AD + BD}{AD} = \frac{AE + CE}{AE}, \text{ or } \frac{AB}{AD} = \frac{AC}{AE}. \qquad (1)$$

In like manner,
$$\frac{AB}{BD} = \frac{AC}{CE}. \qquad (2)$$

Again, applying the theorem of § 220 to the proportions (1) and (2), we obtain the proportions

$$\frac{AB}{AC} = \frac{AD}{AE}, \text{ and } \frac{AB}{AC} = \frac{BD}{CE}.$$

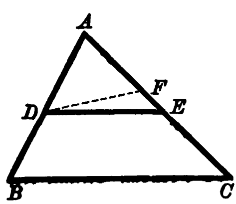
Then by Ax. 1,
$$\frac{AB}{AC} = \frac{AD}{AE} = \frac{BD}{CE}. \qquad (3)$$

The proportions (1), (2), and (3) are all included in the statement of Prop. IX:

A parallel to one side of a triangle divides the other two sides proportionally.

PROP. X. THEOREM

229. (Converse of Prop. IX.) ***A line which divides two sides of a triangle proportionally is parallel to the third side.***



Draw $\triangle ABC$. Draw line DE dividing AB and AC proportionally at D and E, respectively. We then have:

Given, in $\triangle ABC$, line DE meeting AB and AC at D and E, respectively, so that

$$\frac{AB}{AD}=\frac{AC}{AE}.$$

To Prove $DE \parallel BC$.

Proof. 1. If DE is not $\parallel BC$, draw line $DF \parallel BC$, meeting AC at F.

2. Then, $\dfrac{AB}{AD}=\dfrac{AC}{AF}$. (§ 228)

3. But by hyp., $\dfrac{AB}{AD}=\dfrac{AC}{AE}$.

4. Then, $\dfrac{AC}{AE}=\dfrac{AC}{AF}$ or $AE = AF$. (?)

5. Then, line DF coincides with DE, and line DE is $\parallel BC$. (Ax. 3)

Prop. XI. Theorem

230. *In any triangle, the bisector of an angle divides the opposite side into segments proportional to the adjacent sides.*

Draw $\triangle ABC$. Draw line AD, bisecting $\angle A$, meeting BC at D. We then have:

Given line AD bisecting $\angle A$ of $\triangle ABC$, meeting BC at D.

To Prove $\dfrac{DB}{DC}=\dfrac{AB}{AC}$.

Proof. 1. Draw line $BE \parallel AD$, meeting CA extended at E; represent $\angle$s BAD, CAD, and ABE by a, b, c, respectively.

2. We have $\angle c = \angle a$. (?)

3. Also, $\angle E = \angle b$. (?)

4. By hyp., $\angle a = \angle b$.

5. Then, $\angle c = \angle E$. (?)

6. Then, $AB = AE$. (?)

7. Now $\dfrac{DB}{DC}=\dfrac{AE}{AC}$. (§ 228)

8. Then, $\frac{DB}{DC} = \frac{AB}{AC}$. (?)

231. Def. The *segments* of a line by a point are the distances from the point to the extremities of the line, whether the point is in the line itself, or in the line extended.

Ex. 9. Three parallels cut segments 5 and 3, respectively, from a line drawn at random across them. What is the ratio of the segments cut from another line drawn across these three parallels?

? — **Ex. 10.** A line 15 inches long is parallel to the base of a triangle. It divides the sides in the ratio of 3 to 2. Find the base.

PROP. XII. THEOREM

232. *In any triangle the bisector of an exterior angle divides the opposite side externally into segments proportional to the adjacent sides.*

Draw $\triangle ABC$. Extend CA to E and draw line AD bisecting $\angle BAE$, meeting CB extended at D. We then have:

Given line AD bisecting ext. $\angle BAE$ of $\triangle ABC$, meeting CB extended at D.

To Prove $\frac{DB}{DC} = \frac{AB}{AC}$.

(Draw $BF \parallel AD$; then $\angle ABF = \angle AFB$, and $AF = AB$; BF is $\parallel$ to side AD of $\triangle ACD$.)

The above theorem is not true for the exterior angle at the vertex of an isosceles triangle.

Ex. 11. The base of a triangle is 25, and the other sides are 15 and 16, respectively. Draw a line which shall be parallel to the base of the triangle, terminate in the sides, and be 21 in length.

Ex. 12. One of the parallel sides of a trapezoid is double the other. In what ratio do the diagonals intersect?

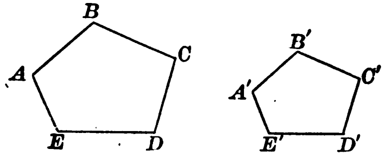
Ex. 13. In a triangle ABC, $AB = 12$, $BC = 15$, $CA = 13$. Find the segments into which the bisector of $\angle B$ divides the opposite side.

Ex. 14. The sides of a triangle are 8, 10, and 12; find the segments of the side 12 made by the bisector of the opposite angle.

Ex. 15. The sides of a triangle are $AB = 6$, $BC = 7$, $CA = 8$. Find the length of the segments into which the bisector of the exterior angle at B divides AC.

SIMILAR POLYGONS

233. Def. Two polygons are said to be *similar* if they are mutually equiangular (§ 121), and have their homologous sides proportional.



Thus, polygons $ABCDE$ and $A'B'C'D'E'$ are similar if

$$\angle A = \angle A', \angle B = \angle B', \text{ etc.},$$

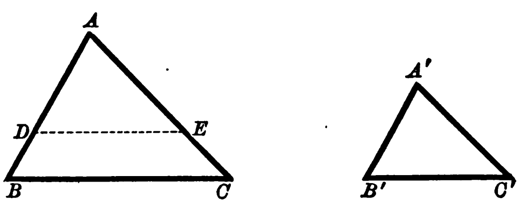
and,
$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

234. The following are given for reference:

1. *In similar polygons, the homologous angles are equal.*
2. *In similar polygons, the homologous sides are proportional.*

PROP. XIII. THEOREM

235. *Two triangles are similar when they are mutually equiangular.*



Draw $\triangle ABC$; also, $\triangle A'B'C'$ having $\angle A' = \angle A$, $\angle B' = \angle B$, and $\angle C' = \angle C$. We then have:

Given, in $\triangle s$ ABC and $A'B'C'$,

$$\angle A = \angle A', \angle B = \angle B', \text{ and } \angle C = \angle C'.$$

To Prove $\triangle$ ABC and $A'B'C'$ similar.

Proof. 1. Place $\triangle A'B'C'$ in position ADE; $\angle A'$ coinciding with $\angle A$, vertices B' and C' falling at D and E, respectively, and side $B'C'$ at DE.

2. We have $$\frac{AB}{AD}=\frac{AC}{AE}.$$ (§ 2

3. Then, $$\frac{AB}{A'B'}=\frac{AC}{A'C'}.$$

4. In like manner, by placing B' at B,

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}.$$

5. Then, $$\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$

6. Then $\triangle$ ABC and $A'B'C'$ have their homologous sides proportional, and are similar. (§ 2

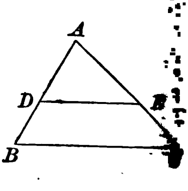
The following are consequences of § 235:

236. *Two triangles are similar when two angles of one equal respectively to two angles of the other.*

For their remaining $\angle$ are equal each to each. (§

237. *Two right triangles are similar when an acute angle of one is equal to an acute angle of the other.*

238. *If a line be drawn between two sides of a triangle parallel to the third side, the triangle formed is similar to the given triangle.*

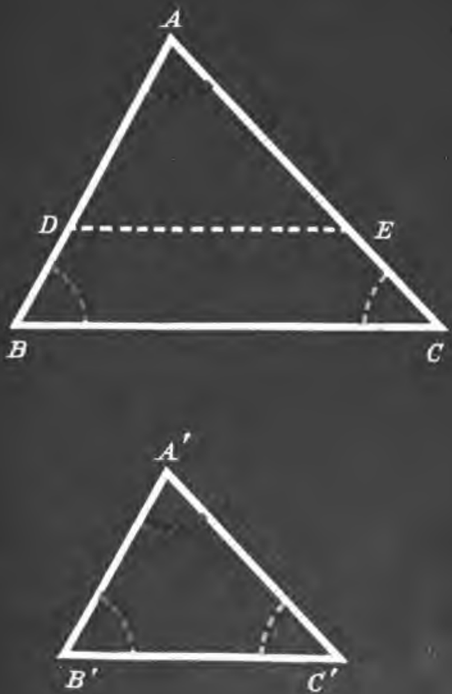


For the triangles are mutually equiangular. (§ 76)

239. Note. *In similar triangles, the homologous sides lie opposite the equal angles.*

Ex. 16. Find the sides of a triangle similar to that in Ex. 14, one side being 9. Does more than one triangle satisfy this condition?

Ex. 17. Each of two isosceles triangles has a vertical angle of 32°. Do their homologous sides form a proportion? Why?



PROP. XIII.

PLATE V

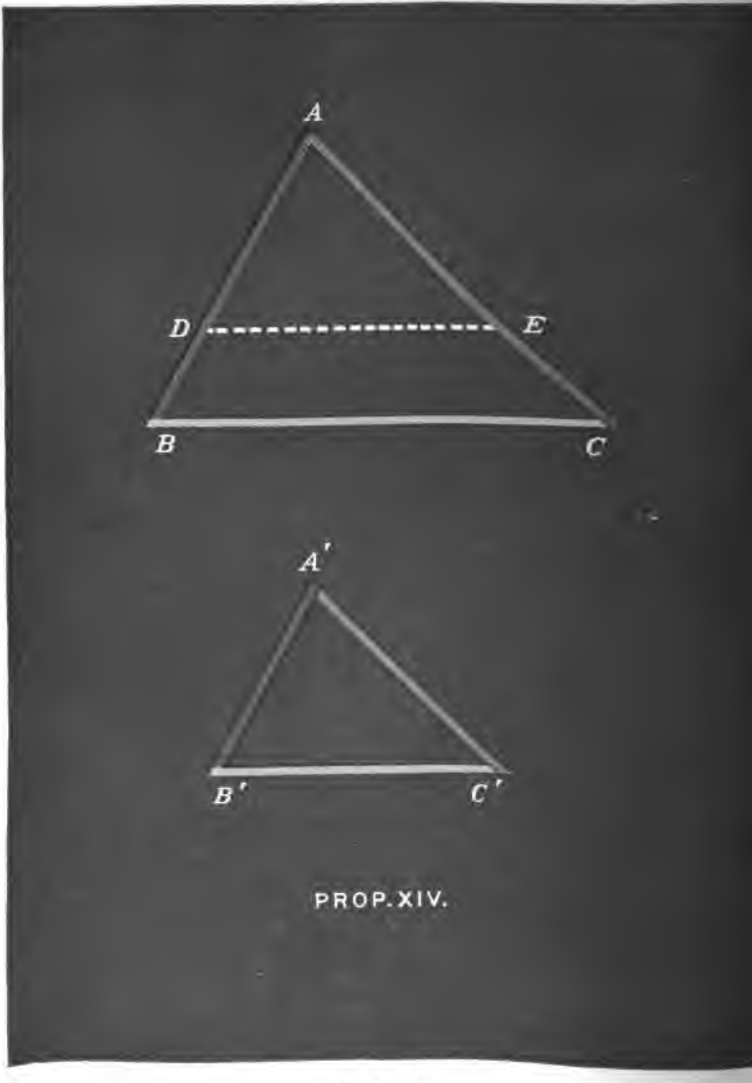
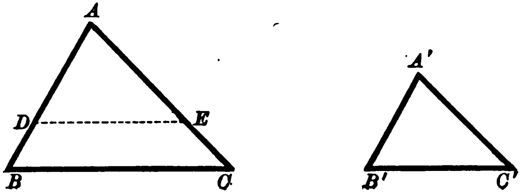


PLATE VI

Prop. XIV. Theorem

240. *Two triangles are similar when their homologous sides are proportional.*



Draw $\triangle$ ABC and $A'B'C'$ having their homologous sides proportional. We then have:

Given, in $\triangle$ ABC and $A'B'C'$,

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}. \quad (1)$$

To Prove $\triangle$ ABC and $A'B'C'$ similar.

Proof. 1. On AB and AC, take $AD = A'B'$ and $AE = A'C'$; and draw line DE.

2. Since $\frac{AB}{AD} = \frac{AC}{AE}$, $DE \parallel BC$. (§ 229)

3. Then $\triangle$ ADE and ABC are similar. (§ 238)

4. Then $\frac{AB}{AD} = \frac{BC}{DE}$, or $\frac{AB}{A'B'} = \frac{BC}{DE}$. (§ 234, 2)

5. Then $DE = B'C'$ by comparing this with (1).

6. Then $\triangle ADE = \triangle A'B'C'$. (?)

7. Then, $\triangle A'B'C'$ is similar to $\triangle ABC$.

Ex. 18. The sides of a triangle are 7, 12, and 14. If the side of a similar triangle homologous to 12 is 18, find the remaining sides of the second triangle.

Ex. 19. The diagonals of any trapezoid divide each other proportionally.

Ex. 20. A post a feet high casts a shadow b feet long. At the same time, a tree casts a shadow c feet long. Find the height of the tree. What theorem of proportion is involved?

241. NOTE. To prove two polygons in general similar, it must be shown that they are mutually equiangular, and have their homologous sides proportional (§ 233); but in the case of two *triangles*, each of these conditions involves the other (§§ 235, 240), so that it is only necessary to show that one of the tests of similarity is satisfied.

PROP. XV. THEOREM

242. *Two triangles are similar when they have an angle of one equal to an angle of the other, and the sides including these angles proportional.*

Draw $\triangle$ ABC and $A'B'C'$ having $\angle A = \angle A'$, and the sides including these sides proportional. We then have:

Given, in $\triangle$ ABC and $A'B'C'$, $\angle A = \angle A'$, and $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$.

To Prove $\triangle$ ABC and $A'B'C'$ similar.

(Place $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle A'$ coincides with $\angle A$ and vertices B' and C' fall at D and E, respectively; by § 229, $DE \parallel BC$; the theorem follows by § 238.)

PROP. XVI. THEOREM

243. *Two triangles are similar when their sides are parallel each to each, or perpendicular each to each.*

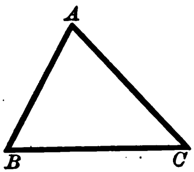


FIG. 1.

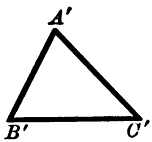


FIG. 2.

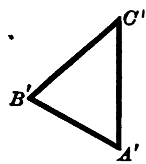


FIG. 3.

Draw $\triangle$ ABC and $A'B'C'$ in accordance with the statement of the proposition. We then have:

Given sides AB, AC, and BC, of $\triangle ABC$, $\parallel$ respectively to sides $A'B'$, $A'C'$, and $B'C'$ of $\triangle A'B'C'$ in Fig. 2, and $\perp$ respectively to sides $A'B'$, $A'C'$, and $B'C'$ of $\triangle A'B'C'$ in Fig. 3.

To Prove $\triangle$ ABC and $A'B'C'$ similar.

Proof. 1. $\angle$s A and A' are either equal or supplementary, as also are B and B', and C and C'. (§§ 81, 82, 83)

2. We may then make the following hypotheses with regard to the $\angle$s of the $\triangle$:

I. $A + A' = 2$ rt. $\angle$s, $B + B' = 2$ rt. $\angle$s, $C + C' = 2$ rt. $\angle$s.

II. $A + A' = 2$ rt. $\angle$s, $B + B' = 2$ rt. $\angle$s, $C = C'$.

III. $A + A' = 2$ rt. $\angle$s, $B = B'$, $C + C' = 2$ rt. $\angle$s.

IV. $A = A'$, $B + B' = 2$ rt. $\angle$s, $C + C' = 2$ rt. $\angle$s.

V. $A = A'$, $B = B'$, whence $C = C'$. (§ 87)

3. The first four hypotheses are impossible; for, in either case, the sum of the $\angle$s of the two $\triangle$ would be $>$ 4 rt. $\angle$s. (§ 84)

4. We then have only $A = A'$, $B = B'$, and $C = C'$.

5. Therefore, $\triangle$ ABC and $A'B'C'$ are similar. (§ 235)

244. Note. 1. *In similar triangles whose sides are parallel each to each, the parallel sides are homologous.*

2. *In similar triangles whose sides are perpendicular each to each, the perpendicular sides are homologous.*

Ex. 21. The bisectors of the equal angles of an isosceles triangle extended to meet the equal sides are divided proportionally.

Ex. 22. The altitudes to the equal sides of an isosceles triangle intersect in a point such that the product of segments of one equals the product of the segments of the other.

Ex. 23. The diagonals of inscribed quadrilateral $ABCD$ intersect at O. If OR and OS are the altitudes of triangles OBC and OAD, respectively, prove

$$\frac{OR}{OS} = \frac{BC}{AD}.$$

Prop. XVII. Theorem

245. *The homologous altitudes of two similar triangles are in the same ratio as any two homologous sides.*

Draw $\triangle$ ABC, and line $AD \perp BC$. Draw $\triangle$ $A'B'C'$, making $\angle B'A'C' = \angle BAC$ and $\angle B' = \angle B$; draw line $A'D' \perp B'C'$. We now have:

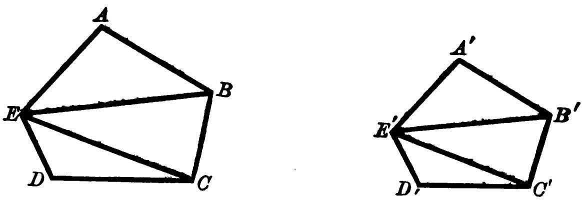
Given AD and $A'D'$ homologous altitudes of similar $\triangle$ ABC and $A'B'C'$.

To Prove $$\frac{AD}{A'D'}=\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$

(Rt. $\triangle$ ABD and $A'B'D'$ are similar by § 237.)

PROP. XVIII. THEOREM

246. *Two polygons are similar when they are composed of the same number of triangles, similar each to each, and similarly placed.*



Draw polygon $ABCDE$; from E draw diagonals EC and EB. Construct $\triangle\ E'D'C'$ similar to $\triangle\ EDC$, sides DC and $D'C'$ being homologous; on $E'C'$, homologous to EC, construct $\triangle\ E'C'B'$ similar to $\triangle\ ECB$; on $E'B'$, homologous to EB, construct $\triangle\ E'B'A'$ similar to $\triangle\ EBA$. We then have:

Given, in polygons AC and $A'C'$, $\triangle\ ABE$ similar to $\triangle\ A'B'E'$, $\triangle\ BCE$ to $\triangle\ B'C'E'$, and $\triangle\ CDE$ to $\triangle\ C'D'E'$.

To Prove polygons AC and $A'C'$ are similar.

Proof. 1. $\triangle$ ABE and $A'B'E'$ are similar; find equal parts of these $\triangle$.

2. $\triangle$ BCE and $B'C'E'$ are similar; find equal parts of these $\triangle$.

3. Now, $\angle\ ABE + \angle\ EBC = \angle\ A'B'E' + \angle\ E'B'C'$, or
$$\angle\ ABC = \angle\ A'B'C'.$$

4. In like manner, $\angle\ BCD = \angle\ B'C'D'$, etc.; and AC and $A'C'$ are mutually equiangular.

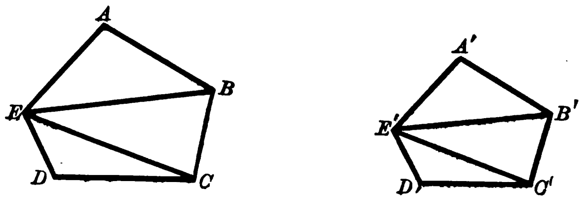
5. Then $\frac{AB}{A'B'}=\frac{BE}{B'E'}$ and $\frac{BE}{B'E'}=\frac{BC}{B'C'}$; whence $\frac{AB}{A'B'}=\frac{BC}{B'C'}$.

6. In like manner, $\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}$, etc.

7. Then AC and $A'C'$ are similar. (§ 233)

Prop. XIX. Theorem

247. (Converse of Prop. XVIII.) *Two similar polygons may be decomposed into the same number of triangles, similar each to each, and similarly placed.*



Draw similar polygons $ABCDE$, $A'B'C'D'E'$, vertices E, E' being homologous; and diagonals EB, EC, $E'B'$, $E'C'$. We then have:

Given E and E' homologous vertices of similar polygons AC and $A'C'$, and lines EB, EC, $E'B'$, and $E'C'$.

To Prove $\triangle ABE$ similar to $\triangle A'B'E'$, $\triangle BCE$ to $\triangle B'C'E'$, and $\triangle CDE$ to $\triangle C'D'E'$.

Proof. 1. In similar polygons, the homologous $\angle$s are equal, and the homologous sides proportional.

2. Then $\triangle$s ABE and $A'B'E'$ are similar. (§ 242)

3. Using other $\angle$s and sides of AC and $A'C'$, and of $\triangle$s ABE and $A'B'E'$, prove $\triangle$s BCE and $B'C'E'$ similar.

4. In like manner, prove $\triangle$s CDE and $C'D'E'$ similar.

Prop. XX. Theorem

248. *The perimeters of two similar polygons are in the same ratio as any two homologous sides.*

Draw similar polygons $ABCDE$, $A'B'C'D'E'$, vertices A, A' being homologous. We then have:

Given AB and $A'B'$, BC and $B'C'$, CD and $C'D'$, etc., homologous sides of similar polygons AC and $A'C'$.

To Prove

$$\frac{AB + BC + CD + \text{etc.}}{A'B' + B'C' + C'D' + \text{etc.}} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

(Apply § 224 to the equal ratios of § 233.)

Ex. 24. From any point in the base of an isosceles triangle perpendiculars to the equal sides are drawn. Prove that these perpendiculars form equal angles with the base and that they are proportional to the segments they cut off from the equal sides; the segments being measured from the extremities of the base towards the vertex.

Ex. 25. A straight line bisecting the exterior angle at the vertex of a triangle is parallel to the base; find the ratio of the sides of the triangle.

Prop. XXI. Theorem

249. *If a perpendicular be drawn from the vertex of the right angle to the hypotenuse of a right triangle,*

I. *The triangles formed are similar to the whole triangle, and to each other.*

II. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

III. *Either leg is a mean proportional between the whole hypotenuse and the adjacent segment.*

Draw $\triangle ABC$, with rt. $\angle$ at C; also, line $CD \perp AB$. We then have:

Given line $CD \perp$ hypotenuse AB of rt. $\triangle ABC$.

To Prove $\triangle$s ACD and BCD similar to $\triangle ABC$, and to each other; also,

$$\frac{AD}{CD}=\frac{CD}{BD}, \frac{AB}{AC}=\frac{AC}{AD}, \text{ and } \frac{AB}{BC}=\frac{BC}{BD}.$$

Proof. 1. Rt. $\triangle$s ACD and ABC are similar. (§ 237)

2. In like manner, $\triangle$s BCD and ABC are similar.

3. Then, $\triangle$s ACD and BCD are similar.

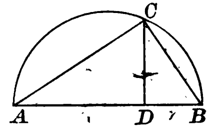
4. In similar $\triangle$s, homologous sides lie opposite equal $\angle$s (§ 239); find sides of $\triangle BCD$ homologous to sides AD and CD of $\triangle ACD$.

5. Then, $\frac{AD}{CD}=\frac{CD}{BD}$. (§ 234, 2)

6. In like manner, from $\triangle$s ABC and ACD, and $\triangle$s ABC and BCD,

$$\frac{AB}{AC}=\frac{AC}{AD}, \text{ and } \frac{AB}{BC}=\frac{BC}{BD}.$$

250. Since an angle inscribed in a semicircle is a right angle (§ 194), it follows that:



If a perpendicular be drawn from any point in the circumference of a circle to a diameter,

1. *The perpendicular is a mean proportional between the segments of the diameter.*

2. *The chord joining the point to either extremity of the diameter is a mean proportional between the whole diameter and the adjacent segment.*

251. We have from the three proportions of § 249,

$$\overline{CD}^2 = AD \times BD,$$

$$\overline{AC}^2 = AB \times AD,$$

and $$\overline{BC}^2 = AB \times BD. \qquad (?)$$

Hence, *if a perpendicular be drawn from the vertex of the right angle to the hypotenuse of a right triangle,*

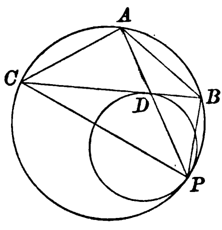
1. *The square of the perpendicular is equal to the product of the segments of the hypotenuse.*

2. *The square of either leg is equal to the product of the whole hypotenuse and the adjacent segment.*

These equations mean that the square of the *numerical measure* of CD equals the product of the *numerical measures* of AD and BD; etc. (Compare § 225.)

Ex. 26. The non-parallel sides AD and BC of trapezoid $ABCD$ intersect at O. If $AB = 15$, $CD = 24$, and the altitude of the trapezoid is 8, what is the altitude of triangle OAB?

Ex. 27. Given two similar triangles, and the altitude of the first double the homologous altitude of the second. Find the ratio between the radii of the circles circumscribing these triangles.



Ex. 28. If two circles are tangent internally, and a chord of the greater is a tangent to the smaller, and if through the points of contact a chord of the greater circle be drawn, then the chords joining the extremities of the tangent and the chord form two pairs of similar triangles.

Ex. 29. The perimeters of two similar polygons are A and B feet, respectively. If a side of the first is a feet, find the homologous side of the second.

Ex. 30. The perimeters of two similar polygons are in the ratio of 4 to 5. If the perimeter of the second is 200, what is the perimeter of the first? If the ratio is A to B and a side of the second is b, what is the homologous side of the first?

Ex. 31. Give a method for stretching strings which mark out a rectangular building for the excavators.

Ex. 32. The legs of a right triangle are 12 and 16. Find the length of the perpendicular from the vertex of the right angle to the hypotenuse.

Ex. 33. In any right triangle if a perpendicular be drawn from the vertex of the right angle to the hypotenuse, the hypotenuse is to either leg as the other leg is to the perpendicular to the hypotenuse.

PROP. XXII. THEOREM

252. *In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.*

Draw $\triangle ABC$ with rt. $\angle$ at C. We then have:

Given $\triangle ABC$, with rt. $\angle$ at C.

To Prove $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

Proof. 1. Draw line $CD \perp AB$,

2. Then, $\overline{AC}^2 = AB \times AD$,

and $\overline{BC}^2 = AB \times BD$. (§ 251, 2)

3. Adding, $\overline{AC}^2 + \overline{BC}^2 = AB \times (AD + BD) = AB \times AB$.

4. Then, $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

253. It follows from § 252 that

$$\overline{AC}^2 = \overline{AB}^2 - \overline{BC}^2, \text{ and } \overline{BC}^2 = \overline{AB}^2 - \overline{AC}^2.$$

That is, *in any right triangle, the square of either leg is equal to the square of the hypotenuse, minus the square of the other leg.*

Ex. 34. If D is the middle point of side BC of right triangle ABC, and DE be drawn perpendicular to the hypotenuse AB, prove

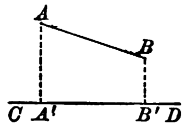
$$\overline{AB}^2 - \overline{CB}^2 = \overline{AE}^2 - \overline{EB}^2.$$

Ex. 35. One leg of a right triangle is double the other. Find the ratio of the segments of the hypotenuse formed by a perpendicular drawn from the vertex of the right angle to the hypotenuse.

Ex. 36. In an isosceles triangle ABC, the altitude CE is extended to D, so that $\angle DAE = 30°$. It is then found that AE is a mean proportional between CE and ED. Find AB in terms of AC.

Ex. 37. If from the sum of the squares of the diagonals of a square we subtract the sum of the squares of the four sides of the square, the result is zero.

254. Defs. The *projection of a point* upon a straight line of indefinite length, is the foot of the perpendicular from the point to the line.



Thus, if line AA' be perpendicular to line CD, the projection of point A on line CD is point A'.

The *projection of a finite straight line* upon a straight line of indefinite length, is that portion of the second line included between the projections of the extremities of the first.

Thus, if lines AA' and BB' be perpendicular to line CD, the projection of line AB upon line CD is line $A'B'$.

Prop. XXIII. Theorem

255. *In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other side upon it.*

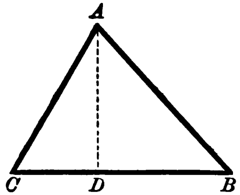


Fig. 1.

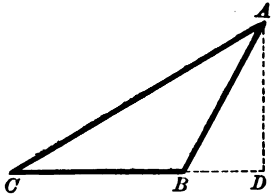


Fig. 2.

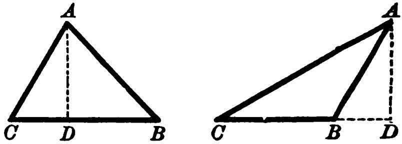
Draw acute-angled $\triangle ABC$; draw also $\triangle ABC$ having an obtuse angle at B. Let CD be projection of CA upon CB. We then have:

Given C an acute $\angle$ of $\triangle ABC$, and CD the projection of side AC upon side CB, extended if necessary. (§ 254)

To Prove $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times CD.$

Proof. 1. Draw line AD; then, $AD \perp CD$. (§ 254)

2. There will be two cases according as D falls on CB (Fig. 1), or on CB extended (Fig. 2); in each figure express BD in terms of BC and CD.



3. Square both members of each of these equations.

4. Add $\overline{AD}^2$ to both members of the result.

5. Find values of $\overline{AD}^2 + \overline{BD}^2$ and $\overline{AD}^2 + \overline{CD}^2$ from the figures. (§ 252)

6. Then, $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times CD.$

Prop. XXIV. Theorem

256. *In any triangle having an obtuse angle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides and the projection of the other side upon it.*

Draw $\triangle ABC$ having an obtuse angle at C; draw $AD \perp BC$, meeting BC extended at D. We then have:

Given C an obtuse $\angle$ of $\triangle ABC$, and CD the projection of side AC upon side BC extended.

To Prove $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2\,BC \times CD.$

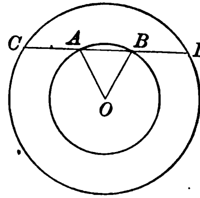
(We have $BD = BC + CD$; square both members, using the algebraic rule for the square of the sum of two numbers, and then add $\overline{AD}^2$ to both members.)

Ex. 38. In any triangle, if a perpendicular be drawn from the vertex to the base, the sum of the other sides of the triangle is to the sum of the segments of the base as the difference of the segments of the base is to the difference of the sides.

Ex. 39. In right triangle ABC, $\overline{BC}^2 = 3\,\overline{AC}^2$. If line CD be drawn from the vertex of the right angle to the middle point of AB, prove $\angle ACD$ equal to 60°.

Ex. 40. If in triangle ABC, $\angle C = 120°$, prove

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + AC \times BC.$$



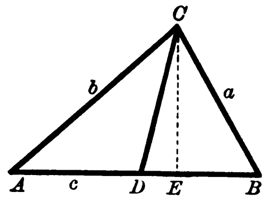
Ex. 41. CD is a chord of the greater of two concentric circles, intersecting the smaller circumference at A and B. CD is 40, $\angle AOB$ is 60°, and AB equals BD; find distance CO and the distance from O to CD.

Prop. XXV. Theorem

257. *In any triangle, if a median be drawn from the vertex to the base,*

I. *The sum of the squares of the other two sides is equal to twice the square of half the base, plus twice the square of the median.*

II. *The difference of the squares of the other two sides is equal to twice the product of the base and the projection of the median upon the base.*



Draw $\triangle ABC$ having $\angle B > \angle A$, and CD a median meeting AB at D. Draw line $CE \perp AB$. We then have :

Given DE the projection of median CD upon base AB (c) of $\triangle ABC$; and AC (b) $>$ BC (a).

To Prove $\quad b^2 + a^2 = 2\,\overline{AD}^2 + 2\,\overline{CD}^2, \qquad (1)$

and $\quad b^2 - a^2 = 2\,c \times DE. \qquad (2)$

Proof. 1. Since $b > a$, E lies between B and D; and $\angle ADC$ is obtuse, and $\angle BDC$ acute.

2. In $\triangle ADC$, $b^2 = \overline{AD}^2 + \overline{CD}^2 + 2\,AD \times DE \qquad$ (§ 256)

$= \overline{AD}^2 + \overline{CD}^2 + c \times DE. \qquad (3)$

3. In $\triangle BDC$, $a^2 = \overline{BD}^2 + \overline{CD}^2 - 2\,BD \times DE \qquad$ (§ 255)

$= \overline{AD}^2 + \overline{CD}^2 - c \times DE. \qquad (4)$

4. Adding and subtracting (3) and (4) gives (1) and (2).

Ex. 42. One side of an inscribed quadrilateral is 14. The diameter of the circle is 24. Find the distance of this side from the centre of the circle correct to three decimal places.

Ex. 43. The radii of two concentric circles are a and b, respectively. Find the length of a chord of the greater which is tangent to the smaller.

Ex. 44. Two unequal circles are tangent externally at A. BC and DE are secants drawn through A, and terminating in one circumference at B and D, in the other at C and E. Chords BD and CE are drawn, forming triangles ABD, ACE. Prove $AB \times AE = AC \times AD$.

Ex. 45. If a median of a triangle is perpendicular to the side, the triangle is isosceles.

Ex. 46. One of the equal sides of an isosceles right triangle is 12 inches in length; find the length of the median drawn to the hypotenuse.

Ex. 47. Two parallel chords are 6 inches apart. The length of the smaller chord is 14 inches, and that of the greater is 32 inches. Find the distance of the smaller chord from the centre.

Ex. 48. If D is the middle point of side BC of triangle ABC, right-angled at C, prove $\overline{AB}^2 - \overline{AD}^2 = 3\,\overline{CD}^2$.

Prop. XXVI. Theorem

258. *If any two chords be drawn through a fixed point within a circle, the product of the segments of one chord is equal to the product of the segments of the other.*

Draw a ⊙; let P be any point within it. Through P draw two chords AB and $A'B'$. We then have:

Given AB and $A'B'$ any two chords passing through fixed point P within ⊙ $AA'B$.

To Prove $AP \times BP = A'P \times B'P$.

Proof. 1. Draw lines AA' and BB'.

2. ∠s A and B' are measured by one-half the same arc, as also are ∠s A' and B. (?)

3. △s $AA'P$ and $BB'P$ are similar. (§ 236)

4. Find sides of △ $BB'P$ homologous to sides AP and $A'P$ of △ $AA'P$. (§ 239)

5. Then, $\dfrac{AP}{A'P} = \dfrac{B'P}{BP}$; and $AP \times BP = A'P \times B'P$. (?)

Note. If, in the figure of Prop. XXVI, chord AB be supposed to revolve about point P as a pivot, the variable segments of the chord will have a constant product; and if one segment increases, the other decreases *in the same ratio.*

If, for example, AP were doubled, BP would be halved.

If two variable magnitudes are so related that, if one increases, the other decreases in the same ratio, they are said to be *reciprocally proportional.*

Thus, the segments of a chord by a fixed point are reciprocally proportional; and the theorem may be written:

If any two chords be drawn through a fixed point within a circle, their segments are reciprocally proportional.

Ex. 49. In a triangle ABC, CD is a median and $4\,\overline{CD}^2 = \overline{AC}^2 + \overline{CB}^2$. Prove that $\overline{AB}^2$ equals $\overline{AC}^2$ plus four times the square of the projection of CD upon CB.

Ex. 50. The sides of a triangle are 21, 20, and 18, respectively. Find the length of the median drawn to the side 18.

Ex. 51. The sides of a triangle are 21, 20, and 18. A median is drawn to the side 18. Find the projection of the median upon this side.

Ex. 52. Using values found in Ex. 50, find the altitude of the above triangle drawn to side 18.

Ex. 53. The median drawn to the base of a triangle is $\frac{13}{2}$. The other sides of the triangle are 12 and 5. Find the base and altitude.

Ex. 54. Given a circle and a chord CD. Find a point P in the circumference such that $\frac{PC}{PD} = \frac{5}{3}$.

Ex. 55. If, in triangle ABC, the altitudes AD, BE, and CF intersect at O, prove $\frac{OA}{OB} = \frac{OE}{OD}$ and $\frac{OB}{OC} = \frac{OF}{OE}$.

Ex. 56. Two sides of a parallelogram are a and b, respectively. Prove that the bisectors of the angles of the parallelogram form a rectangle, whose diagonal is $b - a$.

Ex. 57. Two chords intersect in a circle. The segments of the first are 8 and 9. One segment of the second is $4\frac{1}{2}$; find the other. Also, if the segments of the first are a and b, and a segment of the second is c.

Prop. XXVII. Theorem

259. *If through a fixed point without a circle a secant and a tangent be drawn, the product of the whole secant and its external segment is equal to the square of the tangent.*

Draw a ⊙. Through P, any point without the ⊙, draw line PC tangent to the ⊙ at C; through P draw a secant intersecting the circumference at B, and terminating in the circumference at A. We then have:

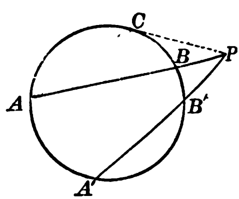
Given AP a secant, and CP a tangent, passing through fixed point P without ⊙ ABC.

To Prove $AP \times BP = \overline{CP}^2$.

(Draw lines AC, BC. $\angle A = \angle BCP$, for each is measured by $\frac{1}{2}$ arc BC (?); then $\triangle$ ACP and BCP are similar, and their homologous sides are proportional.)

260. It follows from § 259 that *if through a fixed point without a circle a secant and a tangent be drawn, the tangent is a mean proportional between the whole secant and its external segment.*

261. *If any two secants be drawn through a fixed point without a circle, the product of one and its external segment is equal to the product of the other and its external segment.*



For if P be any point without ⊙ ABC, AP and $A'P$ secants intersecting the circumference at A and B, and A' and B', respectively, and CP tangent to the ⊙ at C, then

$$AP \times BP = A'P \times B'P;$$

for both $AP \times BP$ and $A'P \times B'P$ equal $\overline{CP}^2$. (§ 259)

Note. If any two secants be drawn through a fixed point without a circle, the entire secants and their external segments are reciprocally proportional. (Compare Note, § 258.)

Ex. 58. Two chords, AB and CD, intersect at K. The centre of the circle is O. $CK = 8$, $KD = 5$, $OK = 3$. Find the radius of the circle.

Ex. 59. A tangent to a circle at a fixed point is 12 feet, and the diameter of the circle is 12 feet. Locate the point from which the tangent is drawn.

Ex. 60. One side of an equilateral triangle is 16; find the altitude, correct to three decimal places.

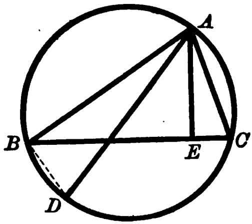
Ex. 61. One side of a square is 1; find the diagonal of the square, correct to three decimal places.

Ex. 62. A tangent 12 inches long and a secant 16 inches long are drawn from an external point to a circle. If the secant passes through the centre, find the radius of the circle.

Ex. 63. If a secant be divided by the circumference in the ratio of 1 to 2, what is the length of the tangent drawn to its extremity?

Prop. XXVIII. Theorem

262. *In any triangle, the product of any two sides is equal to the diameter of the circumscribed circle, multiplied by the perpendicular drawn to the third side from the vertex of the opposite angle.*



Draw $\triangle ABC$. Circumscribe a $\odot$ about it (§ 206); draw diameter AD, and line $AE \perp BC$. We then have:

Given AD a diameter of the circumscribed $\odot$ ACD of $\triangle ABC$, and line $AE \perp BC$.

To Prove $AB \times AC = AD \times AE$.

(In rt. $\triangle$ ABD and ACE, $\angle D = \angle C$; then, the $\triangle$ are similar, and their homologous sides are proportional.)

263. It follows from § 262 that *in any triangle, the diameter of the circumscribed circle is equal to the product of any two sides divided by the perpendicular drawn to the third side from the vertex of the opposite angle.*

Prop. XXIX. Theorem

264. *In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the opposite angle, plus the square of the bisector.*

Draw $\triangle ABC$, and line AD bisecting $\angle A$, and meeting BC at D. We then have:

Given, in $\triangle ABC$, line AD bisecting $\angle A$, meeting side BC at D.

To Prove $AB \times AC = BD \times DC + \overline{AD}^2$.

Proof. 1. Circumscribe a $\odot$ about $\triangle ABC$; extend AD to meet the circumference at E, and draw line CE.

2. In $\triangle$s ABD and ACE, $\angle BAD = \angle CAE$, by hyp.

3. Also $\angle B = \angle E$. (?)

4. Then, $\triangle$s ABD and ACE are similar. (?)

5. Find sides of $\triangle ACE$ homologous to sides AB and AD of $\triangle ABD$. (§ 239)

6. Then, $\frac{AB}{AD} = \frac{AE}{AC}$, or $AB \times AC = AD \times AE$. (?)

7. Then, $AB \times AC = AD \times (DE + AD) = AD \times DE + \overline{AD}^2$.

8. Express $AD \times DE$ in terms of BD and DC. (§ 258)

265. The following cases in which two triangles are *similar* will be found useful in solving original exercises.

Two triangles are similar:

When they have two angles of one equal to two angles of the other (§ 236).

When their homologous sides are proportional (§ 240).

When they have an angle of one equal to an angle of the other, and the sides including these angles proportional (§ 242).

When their sides are parallel each to each, or perpendicular each to each (§ 243).

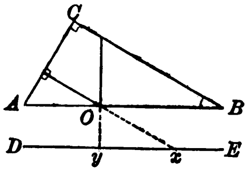
Two right triangles are similar:

When an acute angle of one is equal to an acute angle of the other (§ 237).

Ex. 64. *CD* is a common chord of two equal circles. At *C* tangents are drawn to each circle and meet the circumferences in *E* and *F*. Prove $\triangle EDC$ equal to $\triangle DFC$.

Ex. 65. In a circle whose centre is O, CD is a chord perpendicular to a diameter at any point F not the centre. At C a tangent is drawn, and DE perpendicular to the tangent. Prove $\frac{CD}{OC} = \frac{DE}{CF}$.

Ex. 66. AB is a diameter of a circle with centre at O. CD is a chord perpendicular to AB at E. DF is a line through O, and CF is perpendicular to DF. OC is drawn. Prove $\triangle COE$ similar to $\triangle DFC$.

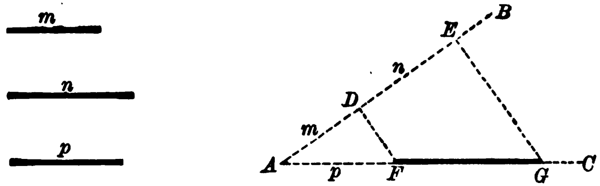


Ex. 67. ABC is a right triangle, O any point in the hypotenuse AB. Through O two perpendiculars are drawn, one to the leg AC, the other to AB. These perpendiculars are extended to meet any line DE parallel to AB, at x and y, respectively. Prove $\frac{xy}{BC} = \frac{Ox}{AB}$.

CONSTRUCTIONS

Prop. XXX. Problem

266. *To construct a fourth proportional* (§ 215) *to three given straight lines.*



Given lines m, n, and p.

Required to construct a fourth proportional to m, n, and p.

Construction. 1. Draw lines AB and AC, making any $\angle$.

2. On AB take $AD = m$, and $DE = n$; on AC take $AF = p$.

3. Draw line DF, and line $EG \parallel DF$, meeting AC at G.

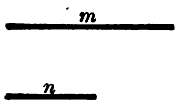
4. Then, $\frac{m}{n} = \frac{p}{FG}$. (?)

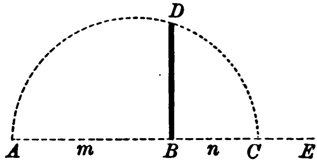
267. If $AF = n$, the above proportion becomes $\frac{m}{n} = \frac{n}{FG}$.

In this case, FG is a *third proportional* (§ 214) to m and n.

Prop. XXXI. Problem

268. *To construct a mean proportional* (§ 214) *between two given straight lines.*





Given lines m and n.

Required to construct a mean proportional between m and n.

Construction. 1. On line AE, take $AB = m$, and $BC = n$.

2. With AC as a diameter, describe a semi-circumference, and draw line $BD \perp AC$, meeting the arc at D.

3. Then, $\dfrac{m}{BD} = \dfrac{BD}{n}$. (§ 250, 1)

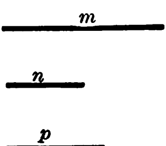
269. By aid of § 268, a line may be constructed equal to $\sqrt{a}$, where a is any number whatever.

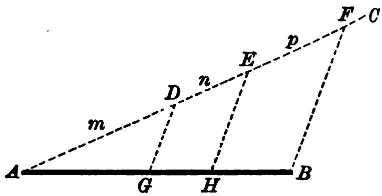
Thus, to construct a line equal to $\sqrt{3}$, we take AB equal to 3 units, and BC equal to 1 unit.

Then, $BD = \sqrt{AB \times BC}$ (§ 217) $= \sqrt{3 \times 1} = \sqrt{3}$.

Prop. XXXII. Problem

270. *To divide a given straight line into parts proportional to any number of given lines.*





Given line AB, and lines m, n, and p.

Required to divide AB into parts proportional to m, n, and p.

Construction. 1. On line AC take $AD = m$, $DE = n$, and $EF = p$.

2. Draw line BF; and lines DG and $EH \parallel BF$, meeting AB at G and H, respectively.

3. Then, $$\frac{AG}{m}=\frac{GH}{n}=\frac{HB}{p}. \quad (1)$$

Proof. 4. In $\triangle AEH$, $$\frac{AH}{AE}=\frac{AG}{m}=\frac{GH}{n}. \quad (?)$$

5. In $\triangle ABF$, $$\frac{AH}{AE}=\frac{HB}{p};$$ whence, equation (1). (?)

Ex. 68. Construct a fourth proportional to three lines whose lengths are 8, 9, and 12, respectively.

Ex. 69. Construct a fourth proportional to 12, 8, and 6.

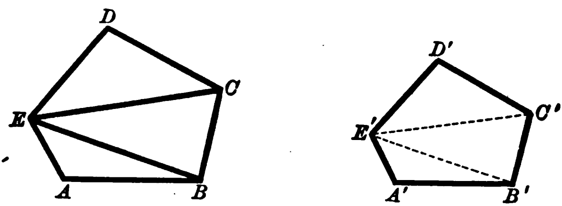
Ex. 70. Construct a third proportional to lines whose lengths are 8 and 6, respectively.

Ex. 71. Construct a mean proportional to lines whose lengths are 9 and 4.

Ex. 72. Construct a line whose length is $\sqrt{5}$.

Prop. XXXIII. Problem

271. *Upon a given side, homologous to a given side of a given polygon, to construct a polygon similar to the given polygon.*



Given polygon $ABCDE$, and line $A'B'$.

Required to construct upon side $A'B'$, homologous to AB, a polygon similar to $ABCDE$.

Construction. 1. Divide polygon $ABCDE$ into $\triangle$s by drawing diagonals EB and EC.

2. Construct $\triangle$s $A'B'E'$, ABE similar, with $\angle A' = \angle A$, and $\angle A'B'E' = \angle ABE$. (?)

3. In like manner, construct $\triangle B'C'E'$ similar to $\triangle BCE$, and $\triangle C'D'E'$ similar to $\triangle CDE$.

272. Def. A straight line is said to be divided by a given point in *extreme and mean ratio* when one of the segments (§ 231) is a mean proportional between the whole line and the other segment.

Thus, line AB is divided *internally* in extreme and mean ratio at C if

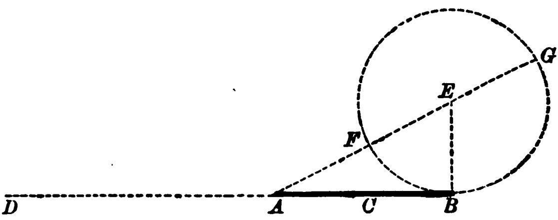
$$\frac{AB}{AC}=\frac{AC}{BC};$$

and *externally* in extreme and mean ratio at D if

$$\frac{AB}{AD}=\frac{AD}{BD}.$$

Prop. XXXIV. Problem

273. *To divide a given straight line in extreme and mean ratio* (§ 272).



Given line AB.

Required to divide it in extreme and mean ratio.

Construction. 1. With radius $BE = \frac{1}{2}\,AB$, draw $\odot BFG$ tangent to AB at B; and line AE cutting circumference at F and G.

2. On AB take $AC = AF$; on BA extended take $AD = AG$; then, AB is divided at C internally, and at D externally, in extreme and mean ratio.

Proof. 3. By § 261, $\frac{AG}{AB}=\frac{AB}{AF}$; or $\frac{AG}{AB}=\frac{AB}{AC}$. (1)

4. Then, $\frac{AG - AB}{AB} = \frac{AB - AC}{AC}$ (?). (2)

5. By cons., $AB = 2\ BE = FG$.

6. Then, $AG - AB = AG - FG = AF = AC$.

7. Substitute in (2), $\frac{AC}{AB} = \frac{BC}{AC}$; or $\frac{AB}{AC} = \frac{AC}{BC}$ (§ 220). (3)

8. Again from (1), $\frac{AG + AB}{AG} = \frac{AB + AC}{AB}$ (?). (4)

9. But $AG + AB = AD + AB = BD$;

and $AB + AC = FG + AF = AD$. (?)

10. Substitute in (4), $\frac{BD}{AD} = \frac{AD}{AB}$; or $\frac{AD}{BD} = \frac{AB}{AD}$.

274. Putting $AB = m$, $AC = x$, $BC = m - x$ in (3),

$$\frac{m}{x} = \frac{x}{m - x}; \text{ or } x^2 = m(m - x). \qquad (\S\ 216)$$

Then, $x^2 = m^2 - mx$; or $x^2 + mx = m^2$.

Multiplying by 4, and adding m^2 to both members,

$$4\,x^2 + 4\,mx + m^2 = 4\,m^2 + m^2 = 5\,m^2.$$

Extracting the square root of both members,

$$2\,x + m = \pm\, m\sqrt{5}.$$

Since x cannot be negative, we take the positive sign before the radical sign; then,

$$2\,x = m\sqrt{5} - m; \text{ and } x \text{ (or } AC) = \frac{m(\sqrt{5} - 1)}{2}.$$

Book IV

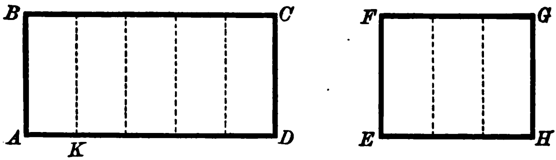
AREAS OF POLYGONS

Prop. I. Theorem

275. *Two rectangles having equal altitudes are to each other as their bases.*

The words "rectangle," "parallelogram," "triangle," etc., in the propositions of Book IV, mean the *amount of surface* in the rectangle, parallelogram, triangle, etc.

Case I. *When the bases are commensurable.*



Draw rectangles $ABCD$ and $EFGH$ having equal altitudes and the bases AD and EH commensurable. We then have:

Given rectangles $ABCD$ and $EFGH$, with equal altitudes AB and EF, and commensurable bases AD and EH.

To Prove $$\frac{ABCD}{EFGH}=\frac{AD}{EH}. \qquad (1)$$

Proof. 1. Let AK be contained 5 times in AD, and 3 times in EH.

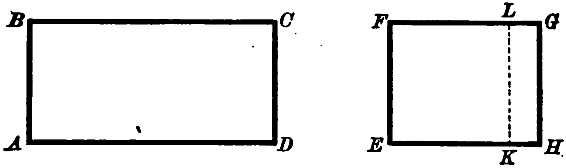
2. Find ratio of AD to EH.

3. Through points of division of AD and EH draw $\perp$s to these lines.

4. Find ratio of $ABCD$ to $EFGH$. (§ 111)

5. Prove equation (1). (?)

Case II. *When the bases are incommensurable.*



Draw rectangles in accordance with the statement. We then have:

Given rectangles $ABCD$ and $EFGH$, with equal altitudes AB and EF, and incommensurable bases AD and EH.

To Prove $$\frac{ABCD}{EFGH} = \frac{AD}{EH}. \qquad (2)$$

Proof. 1. Divide AD into any number of equal parts.

2. Let one of these parts be contained exactly in EK, with a remainder $KH <$ one of the parts.

3. Draw line $KL \perp EH$, meeting FG at L.

4. By Case I, $\dfrac{ABCD}{EFLK} = \dfrac{AD}{EK}$.

5. Let number of subdivisions of AD be indefinitely increased.

6. Find limits $\dfrac{ABCD}{EFLK}$ and $\dfrac{AD}{EK}$.

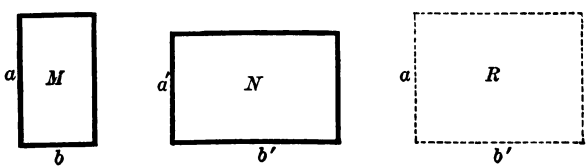
7. Prove equation (2) by Theorem of Limits.

276. Since either side of a rectangle may be taken as the base, it follows that

Two rectangles having equal bases are to each other as their altitudes.

Prop. II. Theorem

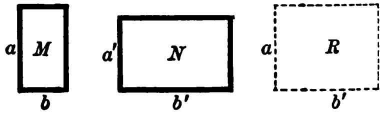
277. *Any two rectangles are to each other as the products of their bases by their altitudes.*



Draw any two **rectangles** M and N. We then have:

Given M and N rectangles, with altitudes a and a', and bases b and b' respectively.

To Prove $\frac{M}{N} = \frac{a \times b}{a' \times b'}$. (1)



Proof. 1. Let R be a rect. with altitude a and base b'.

2. M and R have = altitudes; and bases b and b', respectively; find ratio M to R by § 275.

3. R and N have = bases; and altitudes a and a', respectively; find ratio R to N by § 276.

4. Multiplying results of steps (2) and (3), gives (1).

DEFINITIONS

278. The *area* of a surface is its ratio to another surface, called the *unit of surface*, adopted arbitrarily as the unit of measure (§ 180).

The usual unit of surface is a square whose side is some *linear unit*; for example, a *square inch* or a *square foot*.

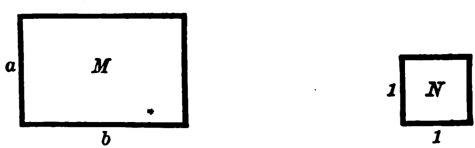
279. We call two surfaces *equivalent* ($\Leftrightarrow$) when their areas are equal.

The *dimensions* of a rectangle are its base and altitude.

Prop. III. Theorem

280. *The area of a rectangle is equal to the product of its base and altitude.*

In all propositions relating to areas, the unit of surface (§ 278) is understood to be *a square whose side is the linear unit.*



Draw any rectangle M, and a square N whose side is 1. We then have:

Given a the altitude and b the base, of rect. M; and N the unit of surface, a square whose side is the linear unit.

To Prove that, if N is the unit of surface,

$$\text{area } M = a \times b. \qquad (1)$$

Proof. 1. We have $\frac{M}{N} = \frac{a \times b}{1 \times 1}$. (§ 277)

2. N being unit of surface, $\frac{M}{N}$ is *area* of M; whence equation (1).

Note. The statement of Prop. III is an abbreviation of the following: If the unit of surface is a square whose side is the linear unit, the *number* which expresses the area of a rectangle is equal to the product of the *numbers* which express the lengths of its sides.

An interpretation of this form is always understood in every proposition relating to areas.

281. By § 280, *the area of a square equals the square of its side.*

Ex. 1. Find the ratio of the area of a square to the product of its diagonals.

Ex. 2. The area of a rectangle whose base is 24 is 432. Find the diagonal of the rectangle.

Ex. 3. How many shingles will it take to cover a roof 36 feet by 18 feet (one side) with shingles that average 4 inches wide, 16 inches long, and are laid 6 inches to the weather?

Prop. IV. Theorem

282. *The area of a parallelogram is equal to the product of its base and altitude.*

Draw ▱ $ABCD$, AD (b) being the base; draw DF (a) the altitude meeting BC at F. We then have:

Given b the base, and a the altitude, of ▱ $ABCD$.

To Prove $\qquad \text{area } ABCD = a \times b. \qquad (1)$

Proof. 1. Draw line $AE \parallel DF$, meeting CB prolonged at E.

2. Then, rt. ▲ ABE and DCF are equal. (§§ 61, 104)

3. If from figure $ADCE$ we take $\triangle\, ABE$, there remains ▱ AC; if we take $\triangle\, DCF$, there remains rect. AF.

4. Then, area $ABCD$ = area $AEFD$; whence equation (1).

283. It follows from § 282 that:

1. *Two parallelograms having equal bases and equal altitudes are equivalent* (§ 279).

2. *Two parallelograms having equal altitudes are to each other as their bases.*

3. *Two parallelograms having equal bases are to each other as their altitudes.*

4. *Any two parallelograms are to each other as the products of their bases by their altitudes.*

Ex. 4. The area of a parallelogram is 288, the base is twice the altitude. Find the dimensions. Construct the parallelogram. Can more than one such parallelogram be drawn? How many and what parts are necessary for a definite figure? Why?

Ex. 5. Find the ratio of the area of a rhombus to the product of its diagonals.

Prop. V. Theorem

284. *The area of a triangle is equal to one-half the product of its base and altitude.*

Draw $\triangle ABC$ having base BC (b); draw $AE \perp BC$, meeting BC, or BC extended, at E. We then have:

Given b the base, and a the altitude of $\triangle ABC$.

To Prove area $ABC = \frac{1}{2}\, a \times b$.

(Draw line $AD \parallel BC$, and line $CD \parallel AB$. By § 105 AC divides $\square\, ABCD$ into two equal $\triangle$s.)

285. It follows from § 284 that:

1. *Two triangles having equal bases and equal altitudes are equivalent.*

2. *Two triangles having equal altitudes are to each other as their bases.*

3. *Two triangles having equal bases are to each other as their altitudes.*

4. *Any two triangles are to each other as the products of their bases by their altitudes.*

5. *The area of any triangle is one-half that of a parallelogram having the same base and altitude.*

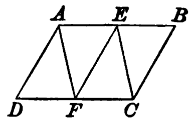
Ex. 6. Draw a line dividing a given right triangle into two equivalent isosceles triangles.

Ex. 7. Prove that each of its medians divides a triangle into two equivalent parts.

Ex. 8. The base of a triangle is 37 feet, the altitude is 32 feet. How many square yards in its area?

Ex. 9. The area of a triangle is 216, the altitude is 12; find the base.

Ex. 10. If E and F are the middle points of sides AB and CD, respectively, of parallelogram $ABCD$, the lines AF, EF, and CE divide the parallelogram into four equal triangles.



Ex. 11. If the middle point of any side of a parallelogram be joined to the opposite vertices, the triangle included by these lines and the opposite side is equivalent to one-half the parallelogram.

Ex. 12. If the middle point of a diagonal of any quadrilateral be joined to the opposite vertices, the quadrilateral is divided into two pairs of equivalent triangles, and into two equivalent parts.

PROP. VI. THEOREM

286. *The area of a trapezoid is equal to one-half the sum of its bases multiplied by its altitude.*

Draw trapezoid $ABCD$, AB (b) and DC (b') being ‖ sides. From D draw altitude DE (a) meeting AB at E. We then have:

Given AB (b) and DC (b') the bases, and DE (a) the altitude, of trapezoid $ABCD$.

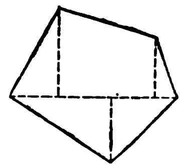
To Prove area $ABCD = a \times \frac{1}{2}(b + b')$.

(Draw diagonal BD. The trapezoid is composed of two $\triangle$ whose altitude is a, and bases b and b', respectively.)

287. Since the line joining the middle points of the non-parallel sides of a trapezoid is equal to one-half the sum of the bases (§ 132), it follows that

The area of a trapezoid is equal to the product of its altitude by the line joining the middle points of its non-parallel sides.

Note. The area of any polygon may be obtained by finding the sum of the areas of the triangles into which the polygon may be divided by drawing diagonals from any one of its vertices.



But in practice it is better to draw the longest diagonal, and draw perpendiculars to it from the remaining vertices of the polygon.

The polygon will then be divided into right triangles and trapezoids; and by measuring the lengths of the perpendiculars, and of the portions of the diagonal which they intercept, the areas of the figures may be found by §§ 284 and 286.

PROP. VII. THEOREM

288. *Two similar triangles are to each other as the squares of their homologous sides.*

Draw $\triangle ABC$; and construct $\triangle A'B'C'$ similar to $\triangle ABC$, AB (c) and $A'B'$ (c') being homologous sides. We then have:

Given c and c' homologous sides of $\triangle$ ABC and $A'B'C'$, respectively.

To Prove $$\frac{ABC}{A'B'C'} = \frac{c^2}{c'^2}. \quad (1)$$

Proof. 1. Draw altitudes $CD(h)$ and $C'D'(h') \perp AB$ and $A'B'$, respectively.

2. Find ratio ABC to $A'B'C'$ in terms of h and c, and h' and c'. (§ 285, 4)

3. Express ratio h to h' in terms of c and c'. (§ 245)

4. Substituting in result of step 2, gives (1).

289. Note. Two similar triangles are to each other as the squares of any two homologous lines.

Ex. 13. A field has two sides parallel, the lengths of these sides being 160 yards and 302 yards, respectively. The distance between the parallel sides is 48 rods. How many acres has the field? Draw a diagram.

Ex. 14. A field has two parallel sides whose lengths are 322 and 168 yards, respectively. The area of the field is 12 acres. Find the distance between the parallel sides and make a diagram of the field.

Ex. 15. A square field contains 3 acres 12 square rods. Find its perimeter in rods, correct to three decimal places.

Ex. 16. A square field contains one-half a square mile. Find its diagonal in rods, correct to three decimal places.

Ex. 17. The base of a rectangle is 462 feet. One diagonal is 504 feet; find the area of the rectangle.

Ex. 18. One-quarter of a government tree claim, containing 160 acres, is to be planted with trees 4 feet apart each way. How many trees will be required? Will the same number of trees be required whether the part selected for planting be rectangular or square?

Ex. 19. The area of a trapezoid is 2774, the altitude is 38, the upper base is 29; find the lower base.

Ex. 20. One of the equal sides of an isosceles trapezoid is 10 feet, the lower base is 12 feet longer than the upper base; find the altitude.

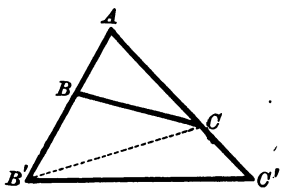
Ex. 21. One side of a rhombus is 20, the longer diagonal is 32; find the other diagonal, the area, and the altitude.

Ex. 22. If from any point within an equilateral triangle perpendiculars to the sides be drawn, the area of the triangle is equal to one-half the sum of the perpendiculars multiplied by one side of the triangle.

Ex. 23. Find the area of a trapezoid whose parallel sides are a and b, and whose altitude is 2, divided by the sum of the parallel sides. If a and b vary, will the area vary? Will the form of the trapezoid change? Draw the trapezoid.

Prop. VIII. Theorem

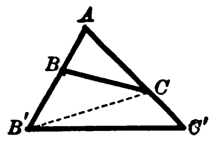
290. *Two triangles having an angle of one equal to an angle of the other, are to each other as the products of the sides including the equal angles.*



Draw $\triangle AB'C'$ and line BC meeting AB' at B, and AC' at C. We then have:

Given $\angle A$ common to $\triangle$s ABC and $AB'C'$.

To Prove $\dfrac{ABC}{AB'C'} = \dfrac{AB \times AC}{AB' \times AC'}$. (1)



Proof. 1. Draw line $B'C$; △ ABC and $AB'C$ have same vertex C, and bases AB and AB' in same str. line.

2. Express ratio ABC to $AB'C$ in terms of AB and AB'. (§ 285, 2)

3. △ $AB'C$ and $AB'C'$ have same vertex B', and bases AC and AC' in same str. line.

4. Express ratio of $AB'C$ to $AB'C'$ in terms of AC and AC'.

5. Multiplying results of steps (2) and (4) gives equation (1).

Ex. 24. If the equal angles of an equilateral triangle be bisected, the bisectors and the base form a triangle whose area is one-third that of the given triangle.

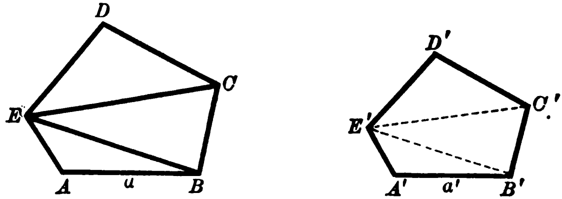
Ex. 25. Any two altitudes of a triangle are inversely as the sides to which they are drawn.

Ex. 26. The altitude of a triangle is 8, its area is 48, and the perpendicular to the base divides the base in the ratio of 2 to 1. Find the sides of the triangle. Construct the triangle.

Ex. 27. Two triangles have equal areas and equal vertical angles. Prove that the products of the sides including the equal angles are equal.

PROP. IX. THEOREM

291. *Two similar polygons are to each other as the squares of their homologous sides.*



Draw polygon $ABCDE$; construct polygon $A'B'C'D'E'$ similar to $ABCDE$, AB (a) and $A'B'$ (a') being homologous sides. We then have:

Given a and a' homologous sides of similar polygons AC and $A'C'$, whose areas are K and K', respectively.

To Prove $$\frac{K}{K'}=\frac{a^2}{a'^2}. \qquad (1)$$

Proof. 1. Draw diagonals from E and E'.

2. $\triangle ABE$ is similar to $\triangle A'B'E'$. (§ 247)

3. Express ratio ABE to $A'B'E'$ in terms of a and a'. (§ 288)

4. Also, $\dfrac{BCE}{B'C'E'}=\dfrac{\overline{BC}^2}{\overline{B'C'}^2}=\dfrac{a^2}{a'^2}$; $\dfrac{CDE}{C'D'E'}=\dfrac{\overline{CD}^2}{\overline{C'D'}^2}=\dfrac{a^2}{a'^2}$. (§ 234, 2)

5. Then, $\dfrac{ABE}{A'B'E'}=\dfrac{BCE}{B'C'E'}=\dfrac{CDE}{C'D'E'}$. (?)

6. Apply § 225 to above result.

7. Substituting values of members in above result gives (1).

Ex. 28. The area of a polygon having a side 6 feet is 112 square feet; find the area of a similar polygon whose homologous side is 9 feet.

Ex. 29. The ratio of the homologous sides of two similar polygons is 2 to 3. What is the ratio of their areas? Find a side of the first. If the area of the second is A, what is the area of the first?

Ex. 30. The area of a polygon is $\frac{1}{1}\frac{5}{6}$ the area of a similar polygon; find the ratio of their homologous sides.

Prop. X. Problem

292. *To find the area of a triangle in terms of its sides.*

Draw $\triangle ABC$, a, b, c, being the sides opposite $\angle s$ A, B, and C, respectively, and $\angle C$ acute. We then have:

Given sides a, b, and c of $\triangle ABC$.

Required to find area ABC in terms of a, b, and c.

Solution. 1. Draw altitude AD.

2. Then, $$c^2=a^2+b^2-2a\times CD. \qquad (§ 255)$$

3. Transposing, $2a\times CD=a^2+b^2-c^2$,

or $$CD=\frac{a^2+b^2-c^2}{2a}.$$

4. Then, $\overline{AD}^2 = \overline{AC}^2 - \overline{CD}^2$ (§ 253)

$$= (AC + CD)(AC - CD)$$

$$= \left(b + \frac{a^2 + b^2 - c^2}{2a}\right)\left(b - \frac{a^2 + b^2 - c^2}{2a}\right)$$

$$= \frac{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)}{4a^2}$$

$$= \frac{[(a+b)^2 - c^2][c^2 - (a-b)^2]}{4a^2}$$

$$= \frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4a^2}. \quad (1)$$

5. Now let $a + b + c = 2s$.

6. Then, $\overline{AD}^2 = \dfrac{2s(2s-2c)(2s-2b)(2s-2a)}{4a^2}$

$$= \frac{16s(s-a)(s-b)(s-c)}{4a^2}.$$

7. Then, $AD = \dfrac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}.$

8. Then, area $ABC = \frac{1}{2}a \times AD$ (?)

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

As an example of § 292, let it be required to find the area of a triangle whose sides are 13, 14, and 15.

Let $a = 13$, $b = 14$, and $c = 15$; then

$$s = \tfrac{1}{2}(13 + 14 + 15) = 21.$$

Whence, $s - a = 8$, $s - b = 7$, and $s - c = 6$.

Then, the area of the triangle is

$$\sqrt{21 \times 8 \times 7 \times 6} = \sqrt{3 \times 7 \times 2^3 \times 7 \times 2 \times 3}$$

$$= \sqrt{2^4 \times 3^2 \times 7^2} = 2^2 \times 3 \times 7 = 84.$$

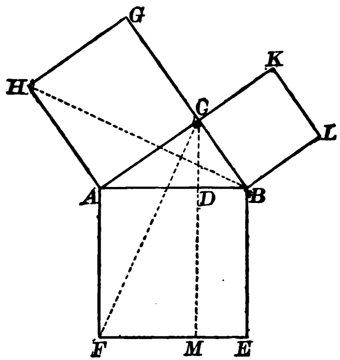
Ex. 31. If a line be drawn from the middle point of each side of a quadrilateral to the middle point of the next side in order, the figure formed is a parallelogram whose area is one-half that of the quadrilateral.

Ex. 32. Prove that the lines connecting the middle points of the sides of a triangle divide it into four equal triangles.

293. Note. Since the area of a square is equal to the square of its side (§ 281), we may state Prop. XXII, Book III, as follows:

In any right triangle, the square described upon the hypotenuse is equivalent to the sum of the squares described upon the legs.

The theorem in the above form may be proved as follows:



Draw $\triangle ABC$, right-angled at C; on AB, AC, and BC describe squares $ABEF$, $ACGH$, and $BCKL$, respectively. We then have:

Given squares AE, AG, and BK described upon the hypotenuse and legs of $\triangle ABC$.

To Prove $ABEF \Leftrightarrow ACGH + BCKL$. (1)

Proof. 1. Draw line $CD \perp AB$, and prolong it to meet EF at M; also, lines BH and CF.

2. We have $\triangle ABH = \triangle ACF$; since $AB = AF$, $AH = AC$, and $\angle BAH = \angle CAF$, each being a rt. $\angle + \angle BAC$. (?)

3. Then $ABH \Leftrightarrow \frac{1}{2}\, ACGH$, having same base and altitude. (§ 285, 5)

4. Again, $ACF \Leftrightarrow \frac{1}{2}\, ADMF$.

5. Then $\frac{1}{2}\, ACGH \Leftrightarrow \frac{1}{2}\, ADMF$, or $ACGH \Leftrightarrow ADMF$.

6. Similarly, drawing lines from A to L and from C to E, $BCKL \Leftrightarrow BDME$.

7. Adding results of (5) and (6), gives equation (1).

Note. The theorem of § 293 is supposed to have been first given by Pythagoras, and is called after him the *Pythagorean Theorem.*

Several other propositions of Book III may be put in the form of statements in regard to areas; for example, Props. XXIII and XXIV.

Ex. 33. In a triangle whose sides are a, b, and c, the angle opposite side c is 90°. Find the area in terms of the sides.

Ex. 34. The sides of a triangle are $a + b$, $b + c$, $c + a$; find the altitude to side $a + b$.

Ex. 35. The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the four sides of the parallelogram.

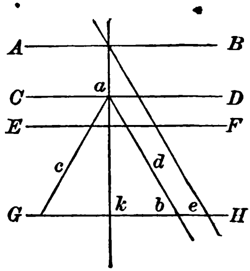
[Project one line upon another.]

Ex. 36. Given a trapezoid with bases b_1 and b_2, altitude h and area S, find the altitude a of an equivalent equilateral triangle whose area is S_2. [$S_1 = S_2$.]

Ex. 37. If lines be drawn through the extremities of the diagonals of a rhombus parallel to the diagonals, they form with the diagonals four rectangles, each equivalent to one-half the rhombus.

Ex. 38. The sides of a rectangle are 24 and 36. The altitude of a triangle of the same area is 33; find the base to which this altitude is drawn. Can more than one triangle be drawn which satisfies the conditions of the problem?

Ex. 39. AB, CD, EF, GH are parallel lines. EF is equidistant from AB and GH. State and prove all theorems suggested by the figure.



Ex. 40. Given two chords AB and AC in a circle of 5-inch radius, such that central angle $AOB = 60°$, and OB is perpendicular to AC, to find the distance of each chord from the centre O.

Ex. 41. If D is the intersection of the perpendiculars from the vertices of triangle ABC to the opposite sides, prove

$$\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2.$$

Ex. 42. Draw figure and give a geometrical proof of the algebraic theorem: The square of the sum of two numbers equals the square of the first, plus twice the product of the first by the second, plus the square of the second.

Ex. 43. Find the ratio of the areas of two triangles which have two sides of one equal to two sides of the other, and the angles formed by the equal sides supplementary.

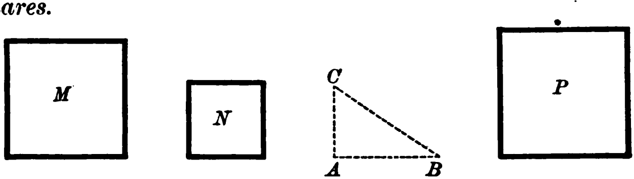
Ex. 44. The square of the altitude of an equilateral triangle is three-fourths the square of one side.

Ex. 45. A person in a train observes that the direction in which the rain appears to fall makes an angle of 60° with the direction of the train's motion, which is at the rate of 45 miles an hour. The rain is really falling vertically. What is its velocity?

CONSTRUCTIONS

PROP. XI. PROBLEM

294. *To construct a square equivalent to the sum of two given squares.*



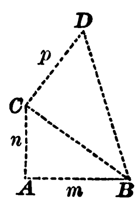
Given squares M and N.

Required to construct a square $\Bumpeq M + N$.

Construction. Construct a rt. $\triangle$ with legs equal to the sides of M and N, and square P with its side equal to the hypotenuse; then, $P \Bumpeq M + N$.

(Prove by § 293.)

295. By an extension of § 293, a square may be constructed equivalent to the sum of any number of given squares.



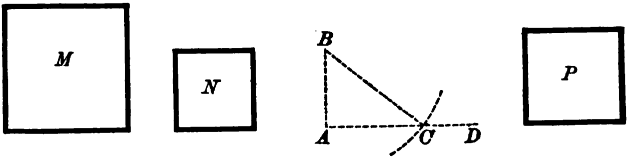
Thus, suppose we have given *three* squares whose sides are m, n, and p, respectively.

Take line $AB = m$; draw line $AC \perp AB$, and equal to n, and line BC; draw line $CD \perp BC$, and equal to p, and line BD.

Then, the square having its side equal to BD will be $\Bumpeq$ the sum of the given squares.

Prop. XII. Problem

296. *To construct a square equivalent to the difference of two given squares.*



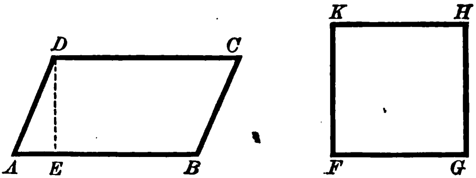
Given squares M and N, $M > N$.

Required to construct a square ⇎ $M - N$.

Construction. Draw rt. $\angle$ BAD, with AB equal to a side of N; find point C in AD, its distance from B equal to a side of M; then, a square with side equal to AC will be ⇎ $M - N$. (Prove by §§ 253 and 281.)

Prop. XIII. Problem

297. *To construct a square equivalent to a given parallelogram.*



Given ▱ $ABCD$.

Required to construct a square ⇎ $ABCD$.

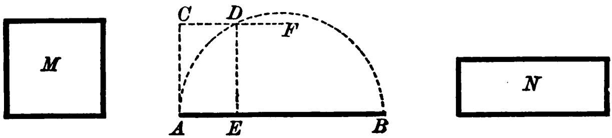
Construction. A square having for its side FG a mean proportional between AB and DE (§ 268), the altitude of the ▱, will be ⇎ $ABCD$. (By § 217, $\overline{FG}^2 = AB \times DE$.)

298. *A square may be constructed equivalent to a given triangle by taking for its side a mean proportional between the base and one-half the altitude of the triangle.*

Ex. 46. To find a point within a parallelogram through which if any line be drawn, the parallelogram is divided into two equivalent parts. Are these parts parallelograms? Trapezoids? Triangles?

PROP. XIV. PROBLEM

299. *To construct a rectangle equivalent to a given square, having the sum of its base and altitude equal to a given line.*



Given square M, and line AB.

Required to construct a rectangle $\Leftrightarrow M$, having the sum of its base and altitude equal to AB.

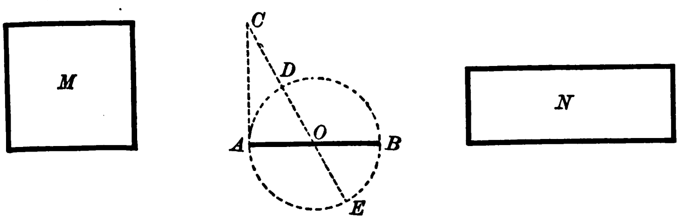
Construction. 1. Construct semi-circumference with diameter AB; and find point D in its arc by drawing $\parallel$ to AB at a distance from AB equal to a side of M.

2. Draw line $DE \perp AB$; then rectangle with dimensions AE and BE will be $\Leftrightarrow M$.

(By § 250, 1, $\frac{AE}{DE} = \frac{DE}{BE}$; and $\overline{DE}^2 = AE \times BE$.)

PROP. XV. PROBLEM

300. *To construct a rectangle equivalent to a given square, having the difference of its base and altitude equal to a given line.*



Given square M, and line AB.

Required to construct a rectangle $\Leftrightarrow M$, having the difference of its base and altitude equal to AB.

Construction. 1. Describe $\odot$ with diameter AB.

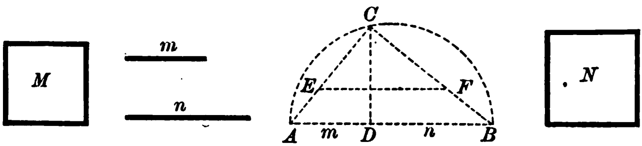
2. Draw tangent AC equal to a side of M.

3. Through C and centre of ⊙ draw a secant, and construct a rectangle with dimensions equal to the secant and its exterior segment.

(By § 259, $CD \times CE = \overline{CA}^2$; also, $AB = CE - CD$.)

Prop. XVI. Problem

301. *To construct a square having a given ratio to a given square.*



Given square M, and lines m and n.

Required to construct a square having to M the ratio $\frac{n}{m}$.

Construction. 1. Take $AD = m$, $BD = n$; with AB as diameter describe a semi-circumference; find point C where ⊥ at D meets arc.

2. On CA take CE equal to a side of M; at E draw ∥ to AB meeting CB at F; square N with side equal to CF will be required square.

Proof. 3. We have $\frac{CF}{CE} = \frac{BC}{AC}$; then $\frac{\overline{CF}^2}{\overline{CE}^2} = \frac{\overline{BC}^2}{\overline{AC}^2}$. (1)

4. In rt. △ ABC, $\frac{\overline{BC}^2}{\overline{AC}^2} = \frac{AB \times BD}{AB \times AD} = \frac{BD}{AD} = \frac{n}{m}$. (§ 250, 2)

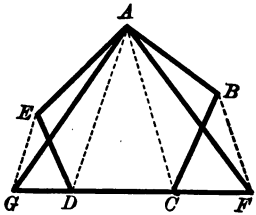
5. Substitute this in (1); and for $\overline{CF}^2$ and $\overline{CE}^2$ areas N and M.

Ex. 47. To inscribe a parallelogram in an equilateral triangle, one angle of the parallelogram being 120°. Can more than one such parallelogram be drawn?

Ex. 48. To inscribe a rhombus in a given triangle, the rhombus and triangle to have one angle in common.

Prop. XVII. Problem

302. *To construct a triangle equivalent to a given polygon.*



Given polygon $ABCDE$.

Required to construct a $\triangle \Leftrightarrow ABCDE$.

Construction. 1. Take three consecutive vertices, A, B, C; draw diagonal AC, and line $BF \parallel AC$, meeting DC prolonged at F, also line AF.

2. $\triangle ACF \Leftrightarrow \triangle ABC$; then, polygon $AFDE \Leftrightarrow ABCDE$ (having common part $ACDE$), and has a number of sides less by 1.

3. Draw diagonal AD, and line $EG \parallel AD$, meeting CD prolonged at G, also line AG; then, $\triangle AED \Leftrightarrow \triangle AGD$, whence $\triangle AFG \Leftrightarrow$ polygon $AFDE$, or to $ABCDE$.

Proof. 4. $\triangle$s ACF and ABC have same base and altitude. (§ 96)

5. Then area $ACF =$ area ABC. (?)

6. Similarly, area $AED =$ area AGD.

Note. By aid of §§ 302 and 298, a square may be constructed equivalent to a given polygon.

Ex. 49. The bases of a trapezoid are 8 and 10, respectively, the altitude 6. Construct an equivalent equilateral triangle.

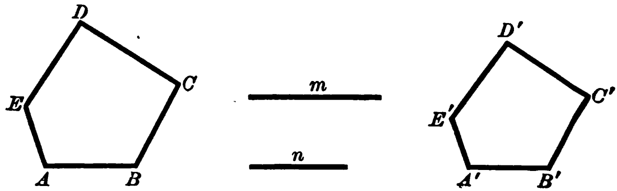
[An equilateral triangle can be constructed when its altitude is known.]

Ex. 50. The bases of a trapezoid are 6 and 8, respectively; the area is 35. Construct the trapezoid and an equivalent equilateral triangle.

Ex. 51. Given $\triangle ABC$. Draw a line from A to BC which shall divide the triangle into two triangles the ratio of whose areas is 1 to 2.

PROP. XVIII. PROBLEM

303. *To construct a polygon similar to a given polygon, and having a given ratio to it.*



Given polygon AC, and lines m and n.

Required to construct a polygon similar to AC, and having to it the ratio $\frac{n}{m}$.

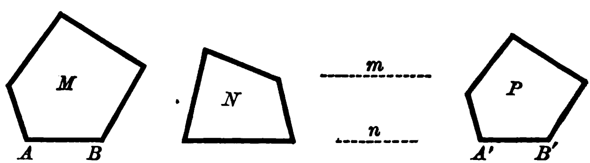
Construction. 1. Let $A'B'$ be side of square having to the square described on AB the required ratio. (§ 301)

2. Polygon $A'C'$, similar to AC, will have required ratio to it.

$\left(\text{By § 291, } \frac{\text{area } A'C'}{\text{area } AC} = \frac{\overline{A'B'}^2}{\overline{AB}^2}, \text{ which equals } \frac{n}{m}.\right)$

PROP. XIX. PROBLEM

304. *To construct a polygon similar to one of two given polygons, and equivalent to the other.*



Given polygons M and N.

Required to construct a polygon similar to M, and $\approx N$.

Construction. 1. Find m and n sides of squares $\approx M$ and N. (Note, § 302)

2. Polygon P, similar to M, with $A'B'$ a fourth proportional to m, n, and AB, will be $\approx N$.

Proof. 3. By § 291, $\dfrac{\text{area } M}{\text{area } P} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2} = \dfrac{m^2}{n^2}$, by cons.

4. Then $\dfrac{\text{area } M}{\text{area } P} = \dfrac{\text{area } M}{\text{area } N}$, and area $P =$ area N.

Ex. 52. To inscribe in a given triangle a parallelogram whose area is one-half the area of the triangle. Can a parallelogram be inscribed whose area is still greater?

Ex. 53. Through a given point P, either within or without a given angle, to draw a line which shall form with the sides of the angle a triangle of given area.

Ex. 54. To construct a triangle whose angles shall be equal, respectively, to the angles of a given triangle, and whose area shall be four times the area of the given triangle.

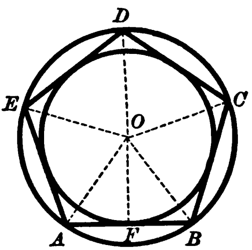
Book V

REGULAR POLYGONS.—MEASUREMENT OF THE CIRCLE.—LOCI

305. Def. A *regular polygon* is a polygon which is both equilateral and equiangular.

Prop. I. Theorem

306. *A circle can be circumscribed about any regular polygon.*



Given regular polygon $ABCDE$.

To Prove that a $\odot$ can be circumscribed about it.

Proof. 1. Draw circumference through A, B, C; also, radii OA, OB, OC, and line OD.

2. In $\triangle$ OAB, OCD, $OB = OC$, $AB = CD$. (?)

3. $\angle OBA = \angle ABC - \angle OBC$, $\angle OCD = \angle BCD - \angle OCB$.

4. Since $\angle ABC = \angle BCD$ (?), and $\angle OBC = \angle OCB$ (?),
$$\angle OBA = \angle OCD.$$

5. Then $\triangle OAB = \triangle OCD$, and $OA = OD$. (?)

6. Then circumference through A, B, C passes through D; and similarly through E.

307. Since AB, BC, CD, etc., are equal chords of the circumscribed ⊙, they are equally distant from O. (§ 164)

Then, a ⊙ drawn with O as a centre, and a line $OF \perp$ to any side AB as a radius, will be *inscribed* in $ABCDE$.

Hence, *a circle can be inscribed in any regular polygon.*

308. Defs. The *centre* of a regular polygon is the common centre of the circumscribed and inscribed circles.

The *angle at the centre* is the angle between the radii drawn to the extremities of any side; as AOB.

The *radius* is the radius of the circumscribed circle, OA.

The *apothem* is the radius of the inscribed circle, OF.

309. From equal ⚠ OAB, OBC, etc., we have

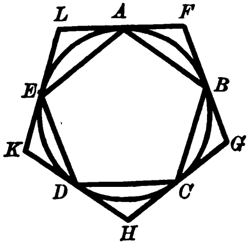
$$\angle AOB = \angle BOC = \angle COD, \text{ etc.} \qquad (?)$$

But the sum of these ∠s is four rt. ∠s. (§ 19)

Whence, *the angle at the centre of a regular polygon is equal to four right angles divided by the number of sides.*

Prop. II. Theorem

310. *If a circumference be divided into equal arcs, their chords form a regular inscribed polygon.*



Given circumference ACD divided into five equal arcs, AB, BC, CD, etc., and chords AB, BC, etc.

To Prove polygon $ABCDE$ regular.

Proof. 1. Sides of polygon are equal by § 159.

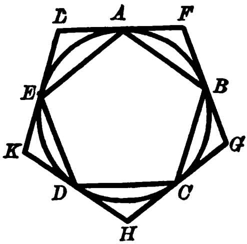
2. Each ∠ is measured by one-half the sum of three of the equal arcs. (§ 192)

311. Let lines LF, FG, etc., be tangent to $\odot\ ACD$ at A, B, etc., respectively, forming polygon $FGHKL$.

In $\triangle\ ABF$, BCG, CDH, etc., $AB = BC = CD$, etc. (§ 159)

Also, since arc AB = arc BC = arc CD, etc., we have

$$\angle BAF = \angle ABF = \angle CBG = \angle BCG, \text{ etc.} \quad (§\ 196)$$



Whence, ABF, BCG, etc., are equal isosceles $\triangle$. (§§ 49, 90)

Then, $\angle F = \angle G = \angle H$, etc.,

and $BF = BG = CG = CH$, etc. (§ 48)

Then, $FG = GH = HK$, etc.,

and polygon $FGHKL$ is regular.

Then, *if a circumference be divided into equal arcs, tangents at the points of division form a regular circumscribed polygon.*

312. It follows from §§ 310 and 311 that

1. *If from the middle point of each arc subtended by a side of a regular inscribed polygon lines be drawn to its extremities, a regular inscribed polygon of double the number of sides is formed.*

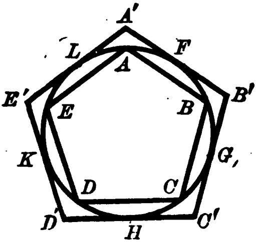
2. *If at the middle point of each arc included between two consecutive points of contact of a regular circumscribed polygon tangents be drawn, a regular circumscribed polygon of double the number of sides is formed.*

3. *An equilateral polygon inscribed in a circle is regular.*

For its sides subtend equal arcs. (?)

PROP. III. THEOREM

313. *Tangents to a circle at the middle points of the arcs subtended by the sides of a regular inscribed polygon, form a regular circumscribed polygon.*



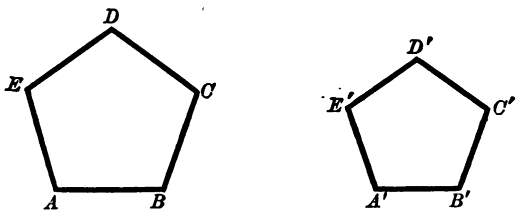
Given $ABCDE$ a regular polygon inscribed in $\odot AC$, and polygon $A'B'C'D'E'$ with sides $A'B'$, $B'C'$, etc., tangent to $\odot AC$ at middle points F, G, etc., of arcs AB, BC, etc., respectively.

To Prove $A'B'C'D'E'$ a regular polygon.

(Arc $AF =$ arc $BF =$ arc $BG =$ arc CG, etc.; use § 311.)

PROP. IV. THEOREM

314. *Regular polygons of the same number of sides are similar.*



Given AC and $A'C'$ regular polygons of five sides.

To Prove AC and $A'C'$ similar. (Use § 233.)

Find the angle, and the angle at the centre,

Ex. 1. Of a regular pentagon.

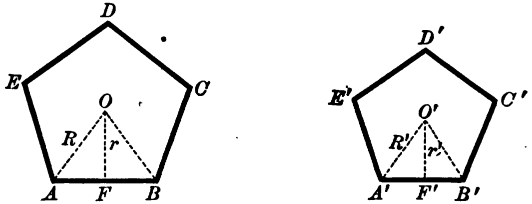
Ex. 2. Of a regular dodecagon.

Ex. 3. Of a regular polygon of 32 sides.

Ex. 4. Of a regular polygon of 25 sides.

PROP. V. THEOREM

315. *The perimeters of two similar regular polygons are to each other as their radii, or as their apothems.*



Given O, O' the centres, OA (R), $O'A'$ (R') the radii, OF (r), $O'F'$ (r') the apothems, and P, P' the perimeters, respectively, of similar regular polygons AC and $A'C'$.

To Prove $$\frac{P}{P'}=\frac{R}{R'}=\frac{r}{r'}. \qquad (1)$$

Proof. 1. Draw lines OB, $O'B'$; then $\angle AOB = \angle A'O'B'$. (§ 309)

2. Since $\frac{R}{R'}=\frac{OB}{O'B'}$, $\triangle$s OAB, $O'A'B'$ are similar. (§ 242)

3. Then, $$\frac{AB}{A'B'}=\frac{R}{R'}=\frac{r}{r'}. \qquad (§ 245)$$

4. Now $$\frac{AB}{A'B'}=\frac{P}{P'}. \qquad (§ 248)$$

5. Substituting in result of (3) gives equation (1).

316. If K denote the area of polygon AC, and K' of $A'C'$,

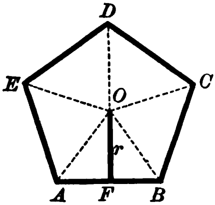
$$\frac{K}{K'}=\frac{\overline{AB}^2}{\overline{A'B'}^2}. \qquad (§ 291)$$

But, $\frac{AB}{A'B'}=\frac{R}{R'}=\frac{r}{r'}$; whence, $\frac{K}{K'}=\frac{R^2}{R'^2}=\frac{r^2}{r'^2}$.

That is, *the areas of two similar regular polygons are to each other as the squares of their radii, or as the squares of their apothems.*

PROP. VI. THEOREM

317. *The area of a regular polygon is equal to one-half the product of its perimeter and apothem.*



Given P the perimeter, and r the apothem, of regular polygon AC.

To Prove area $AC = \frac{1}{2} P \times r$.

($\triangle$ OAB, OBC, etc., have common altitude r.)

Ex. 5. One side of a regular hexagon is $2\sqrt{3}$. Find the radius of the inscribed circle.

Ex. 6. The apothem of a regular polygon is 9, and the area 182. What is the perimeter?

Ex. 7. In an equilateral triangle find the area K in terms of the apothem r.

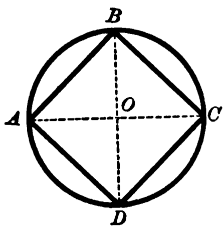
Ex. 8. In a regular hexagon find the apothem in terms of R; find the area in terms of R; find the area in terms of the apothem.

Ex. 9. Find the ratio of the altitude and apothem of an equilateral triangle.

Ex. 10. A triangle whose area is 120 is inscribed in a semicircle whose radius is 13. Construct the triangle.

PROP. VII. PROBLEM

318. *To inscribe a square in a given circle.*



Given $\odot$ AC.

Required to inscribe a square in $\odot$ AC.

Construction. Draw $\perp$ diameters AC and BD, and lines AB, BC, CD, and DA; $ABCD$ is a square.

(The proof is left to the pupil; see § 310.)

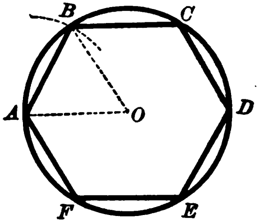
319. Denoting radius OA by R,

$$\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 = 2R^2 \text{ (§ 252); whence, } AB = R\sqrt{2}.$$

That is, *the side of an inscribed square is equal to the radius of the circle multiplied by* $\sqrt{2}$.

Prop. VIII. Problem

320. *To inscribe a regular hexagon in a given circle.*



Given $\odot$ AC.

Required to inscribe a regular hexagon in $\odot$ AC.

Construction. 1. Find point B in circumference at a distance from A equal to radius OA.

2. Line AB is a side of a regular inscribed hexagon; and to inscribe a regular hexagon in a $\odot$ apply radius 6 times as a chord.

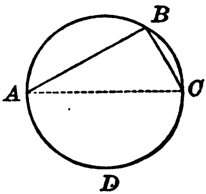
Proof. 3. $\triangle AOB$ is equiangular. (§ 51)

4. Then $\angle AOB$ is $\frac{1}{3}$ of 2 rt. $\angle$s (?), or $\frac{1}{6}$ of 4 rt. $\angle$s.

321. It follows from § 320 that *the side of a regular inscribed hexagon equals the radius of the circle.*

322. Note. If chords be drawn joining the alternate vertices of a regular inscribed hexagon, there is formed an inscribed equilateral triangle.

323. Let AB be a side of an equilateral $\triangle$ inscribed in $\odot$ AD, whose radius is R.



Drawing diameter AC, and chord BC, BC is a side of a regular inscribed hexagon, and therefore equal to R. (§§ 321, 322)

Now ABC is a rt. $\triangle$. (§ 194)

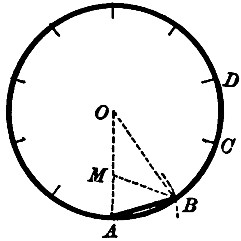
Then by § 253, $\overline{AB}^2 = \overline{AC}^2 - \overline{BC}^2 = (2\,R)^2 - R^2 = 3\,R^2$.

Whence, $AB = R\sqrt{3}$.

Then, *the side of an inscribed equilateral triangle equals the radius of the circle multiplied by* $\sqrt{3}$.

PROP. IX. PROBLEM

324. *To inscribe a regular decagon in a given circle.*



Given $\odot AC$.

Required to inscribe a regular decagon in $\odot AC$.

Construction. 1. Divide radius OA internally in extreme and mean ratio (§ 273), making

$$\frac{OA}{OM}=\frac{OM}{AM}. \qquad (1)$$

2. OM is a side of a regular inscribed decagon.

Proof. 3. Take chord $AB = OM$, and draw lines OB, BM; in $\triangle$ OAB, ABM, $\angle A = \angle A$.

4. Since $OM = AB$, (1) becomes $\dfrac{OA}{AB}=\dfrac{AB}{AM}$.

5. $\triangle$ OAB, ABM are similar (§ 242); then $\angle ABM = \angle O$.

6. $\triangle ABM$ is isosceles, being similar to $\triangle OAB$.

7. By Ax. 1, $AB = BM = OM$; then $\angle OBM = \angle O$. (?)

8. Add results of (5) and (7), $\angle OBA = 2\angle O$. (2)

9. Find sum of $\angle$s of $\triangle OAB$; $2\angle OBA + \angle O = 180°$. (?)

10. Substitute in this value of $\angle OBA$ in (2), then

$$5\angle O = 180°; \text{ and } \angle O \text{ is } \tfrac{1}{10} \text{ of 4 rt. } \angle s.$$

325. Note. If chords be drawn joining the alternate vertices of a regular inscribed decagon, there is formed a regular inscribed pentagon.

326. Denoting the radius of the $\odot$ by R, we have

$$AB = OM = \frac{R(\sqrt{5}-1)}{2}. \qquad (\S\ 274)$$

This is an expression for the side of a regular inscribed decagon in terms of the radius of the circle.

Ex. 11. The diameter of a circle is 20. Inscribe a regular hexagon and find its area.

Ex. 12. The side of a regular hexagon is 6; find its area. If this hexagon be an inscribed one, what is the area of the regular hexagon circumscribing the same circle?

Ex. 13. A regular polygon is inscribed in a circle. Give a general method for finding its area.

Ex. 14. An equilateral triangle is inscribed in a circle. If its side is 10, what is its area, and how does it compare with the area of the equilateral triangle circumscribing the same circle?

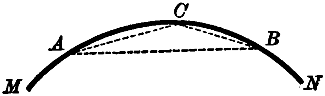
Ex. 15. If the altitude of the triangle in exercise 14 were 10, what would be its area and the area of the equilateral triangle circumscribing the circle?

Ex. 16. The centre of a regular hexagon bisects every line drawn through it which terminates in the sides of the hexagon.

Ex. 17. The apothem of an inscribed equilateral triangle is one-half the radius of the circle.

Prop. X. Problem

327. *To construct the side of a regular pentedecagon inscribed in a given circle.*



Given arc MN.

Required to construct the side of a regular inscribed polygon of fifteen sides.

Construction. If AB is a side of a regular inscribed hexagon (§ 321), and AC a side of a regular inscribed decagon (§ 326), arc BC is $\frac{1}{6}-\frac{1}{10}$, or $\frac{1}{15}$ of the circumference.

Ex. 18. An equilateral triangle is inscribed in a circle. The distance from the intersection of the medians to the middle point of the base is 6. Find the radius of the circle.

Ex. 19. If a circle be inscribed in an equilateral triangle, the altitude of the triangle passes through the centre of the circle, and is three times the radius of the circle.

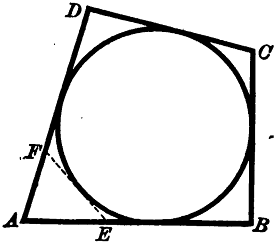
Ex. 20. The area of the square which circumscribes a circle is twice that of the inscribed square.

Ex. 21. The area of the equilateral triangle circumscribing a circle is four times the area of the inscribed one.

MEASUREMENT OF THE CIRCLE

Prop. XI. Theorem

328. *The circumference of a circle is shorter than the perimeter of any circumscribed polygon.*



Given polygon $ABCD$ circumscribed about a ⊙.

To Prove circumference of ⊙ shorter than perimeter $ABCD$.

Proof. 1. Of the perimeters of the ⊙, and of all possible circumscribed polygons, there must be some perimeter such that all the others are of the same or greater length.

2. But no circumscribed polygon can have this perimeter; for suppose polygon $ABCD$ to have this perimeter, and draw line EF tangent to the ⊙, meeting AB at E and AD at F.

3. We know that EF is $< (AE + AF)$. (Ax. 6)

4. Then, the perimeter of polygon $BEFDC$ is $<$ the perimeter of polygon $ABCD$.

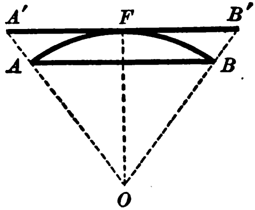
5. Hence, the circumference of the ⊙ is $<$ the perimeter of any circumscribed polygon.

PROP. XII. THEOREM

329. *If a regular polygon be inscribed in, or circumscribed about, a circle, and the number of its sides be indefinitely increased,*

I. *Its perimeter approaches the circumference as a limit.*

II. *Its area approaches the area of the circle as a limit.*



Given p, P perimeters, k, K areas, of two regular polygons of same number of sides, respectively inscribed in, and circumscribed about, a ⊙, whose circumference is C, and area S.

To Prove that, if the number of sides of the polygons be indefinitely increased, P and p approach the limit C, and K and k the limit S.

Proof. 1. Let $A'B'$ be side of polygon whose perimeter is P; draw radius OF to point of contact.

2. If OA' and OB' cut circumference at A and B, AB is side of polygon whose perimeter is p. (§ 309)

3. Polygons are similar; then $\frac{P}{p} = \frac{OA'}{OF}$. (§§ 314, 315)

4. Then,

$$\frac{P-p}{p} = \frac{OA' - OF}{OF} \text{ (?)}; \text{ or } P - p = \frac{p}{OF} \times (OA' - OF).$$

5. p is $<$ circumference; $OA' - OF$ is $< A'F$. (Ax. 7, § 62)

6. Then, $$P - p < \frac{C}{OF} \times A'F. \qquad (1)$$

7. If number of sides of each polygon be indefinitely increased, the polygons continuing to have same number of sides, the length of each side will be indefinitely diminished, and $A'F$ will approach the limit 0.

8. $\frac{C}{OF}$ being constant, $P - p$ will approach limit 0.

9. Now circumference is $< P$, and $> p$. (§ 328, Ax. 7)

10. Then, $P - C$ and $C - p$ will approach limit 0, and P and p will approach limit C.

11. Again, by § 316, $\frac{K}{k} = \frac{\overline{OA'}^2}{\overline{OF}^2}$; or $\frac{K-k}{k} = \frac{\overline{OA'}^2 - \overline{OF}^2}{\overline{OF}^2}$. (?)

12. Then, $$K - k = \frac{k}{\overline{OF}^2} \times \overline{A'F}^2.$$ (§ 253)

13. If the number of sides of each polygon be indefinitely increased, the polygons continuing to have the same number of sides, $A'F$ will approach limit 0.

14. $\frac{k}{\overline{OF}^2} \times \overline{A'F}^2$ being $< \frac{S}{\overline{OF}^2} \times \overline{A'F}^2$, will approach limit 0.

15. Then, $K - k$ will approach limit 0; and S being $< K$, and $> k$, K and k will approach limit S.

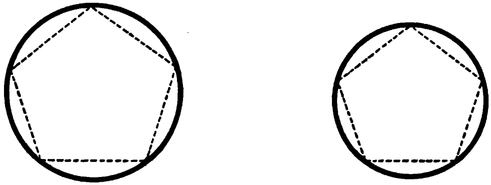
330. *If a regular polygon be inscribed in a circle, and the number of its sides be indefinitely increased, its apothem approaches the radius of the circle as a limit.*

For, it was shown in § 329 that $OA' - OF$ approaches the limit 0, whence OF approaches the limit OA'.

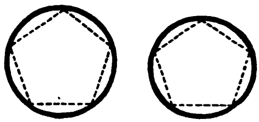
OF is the apothem of a regular polygon inscribed in a circle whose radius is OA'.

PROP. XIII. THEOREM

331. *The circumferences of two circles are to each other as their radii.*



Given C and C' the circumferences of two ⓢ whose radii are R and R', respectively.



To Prove $\dfrac{C}{C'}=\dfrac{R}{R'}$. (1)

Proof. 1. Inscribe similar regular polygons whose perimeters are P and P', in ⓢ whose radii are R and R'.

2. Then $\dfrac{P}{P'}=\dfrac{R}{R'}$ (§ 315); whence $P\times R'=P'\times R$.

3. If number of sides of polygons be indefinitely increased, $P\times R'$ approaches limit $C\times R'$, and $P'\times R$ the limit $C'\times R$. (§ 329, I)

4. Equate these limits (§ 187), and use § 218.

332. Multiplying the terms of the ratio $\dfrac{R}{R'}$ by 2, we have

$$\frac{C}{C'}=\frac{2R}{2R'}=\frac{D}{D'},$$

if D and D' denote the diameters of the ⓢ whose radii are R and R', respectively.

That is, *the circumferences of two circles are to each other as their diameters.*

333. The proportion $\dfrac{C}{C'}=\dfrac{D}{D'}$ (§ 332) may be written

$$\frac{C}{D}=\frac{C'}{D'}. \qquad (§ 220)$$

That is, *the ratio of the circumference of a circle to its diameter has the same value for every circle.*

This constant value is denoted by the symbol π; then,

$$\frac{C}{D}=\pi. \qquad (1)$$

It is shown by methods of higher mathematics that the ratio π is incommensurable; its numerical value can only be obtained approximately.

Its value to the nearest fourth decimal place is 3.1416.

334. Equation (1) of § 333 gives

$$C=\pi D.$$

That is, *the circumference of a circle is equal to its diameter multiplied by* π.

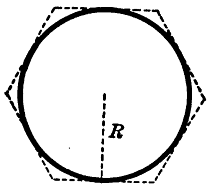
We also have $C = 2\pi R$.

That is, *the circumference of a circle is equal to its radius multiplied by* 2π.

335. Def. In circles of different radii, *similar arcs, similar segments*, and *similar sectors* are those which correspond to equal central angles.

PROP. XIV. THEOREM

336. *The area of a circle is equal to one-half the product of its circumference and radius.*



Given R the radius, C the circumference, and S the area of a $\odot$.

To Prove $S = \frac{1}{2} C \times R.$ (1)

Proof. 1. Circumscribe a regular polygon about the $\odot$; let P denote its perimeter, and K its area.

2. We have $K = \frac{1}{2} P \times R.$ (§ 317)

3. If number of sides of polygon be indefinitely increased, K approaches limit S, and $\frac{1}{2} P \times R$ the limit $\frac{1}{2} C \times R.$ (§ 329)

4. Equating limits (?) gives equation (1).

337. We have $C = 2\pi R$; then, $S = \pi R \times R = \pi R^2$.

That is, *the area of a circle is equal to the square of its radius multiplied by* π.

Again, $S = \frac{1}{4}\pi \times 4R^2 = \frac{1}{4}\pi \times (2R)^2$.

If D denote the diameter of the $\odot$, $S = \frac{1}{4}\pi D^2$.

That is, *the area of a circle is equal to the square of its diameter multiplied by* $\frac{1}{4}\pi$.

338. Let S and S' denote the areas of two ⓢ whose radii are R and R', and diameters D and D', respectively.

Then, $$\frac{S}{S'}=\frac{\pi R^2}{\pi R'^2}=\frac{R^2}{R'^2},$$

and $$\frac{S}{S'}=\frac{\frac{1}{4}\pi D^2}{\frac{1}{4}\pi D'^2}=\frac{D^2}{D'^2}. \qquad (\S\ 337)$$

That is, *the areas of two circles are to each other as the squares of their radii, or as the squares of their diameters.*

339. Let s be the area, and c the arc, of a sector of a ⊙, whose area is S, circumference C, and radius R.

Since a sector is the same part of the ⊙ that its arc is of the circumference,

$$\frac{s}{S}=\frac{c}{C}, \text{ or } s=c\times\frac{S}{C}.$$

Then, by § 336, $s=c\times\frac{1}{2}R=\frac{1}{2}c\times R.$

Hence, *the area of a sector equals one-half the product of its arc and radius.*

Since similar sectors are like parts of the ⓢ to which they belong (§ 335), it follows that

Similar sectors are to each other as the squares of their radii.

Ex. 22. The area of a circle is 64π; find the circumference.

Ex. 23. Give geometric method for cutting the largest possible octagon from a board one foot square.

Ex. 24. The area of a circle is 154, find its circumference and radius.

Ex. 25. The area of a circle is 144π. Find the area of an inscribed equilateral triangle.

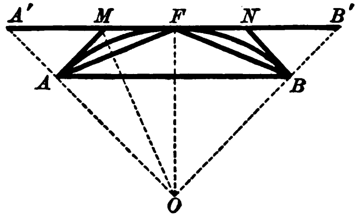
Ex. 26. The radius of a circle is 14; find its area. If the radius were doubled, how would the area be affected?

Ex. 27. What is the area of the largest circle which can be cut from a triangular piece of cardboard whose edges are 5, 13, and 12 inches, respectively?

Ex. 28. The ratio of the areas of two circles is $\frac{9}{16}$, one circumference is 37.6992; find the other circumference. Will more than one circumference satisfy the given conditions? Why? ($\pi \doteq 3.1416$.)

Prop. XV. Problem

340. *Given p and P, the perimeters of a regular inscribed and of a regular circumscribed polygon of the same number of sides, to find p' and P', the perimeters of a regular inscribed and of a regular circumscribed polygon having double the number of sides.*



Solution. 1. Let AB be side of polygon whose perimeter is p.

2. Let radius OF bisect arc AB at F; let radii OA and OB extended cut tangent at F at A' and B', respectively; then $A'B'$ is side of polygon whose perimeter is P. (§ 309)

3. Draw chords AF, BF, and tangents AM, BN, meeting $A'B'$ at M and N, respectively; then AF and MN are sides of polygons whose perimeters are p' and P'. (§ 312)

4. If n denotes number of sides of polygons whose perimeters are p and P, and $2n$ number of sides of polygons whose perimeters are p' and P',

$$AB = \frac{p}{n},\ A'B' = \frac{P}{n},\ AF = \frac{p'}{2n},\ MN = \frac{P'}{2n}. \qquad (1)$$

5. Line OM bisects $\angle A'OF$; whence, $\dfrac{A'M}{MF} = \dfrac{OA'}{OF}$. (§§ 175, 230)

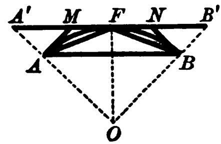
6. Again, $$\frac{P}{p} = \frac{OA'}{OF}. \qquad (§ 315)$$

7. Then, $$\frac{P}{p} = \frac{A'M}{MF},\ \text{and}\ \frac{P+p}{p} = \frac{A'M + MF}{MF}. \qquad (?)$$

8. Then, $$\frac{P+p}{2p} = \frac{A'F}{2\,MF} = \frac{\frac{1}{2}A'B'}{MN} = \frac{P}{2n} \div \frac{P'}{2n} = \frac{P}{P'}.$$

9. Then, $P' \times (P+p) = 2\,P \times p$, or $P' = \dfrac{2\,P+p}{P+p}$. (2)

10. Again, since $\angle ABF = \angle AFM$, $\triangle$ ABF, AFM are similar. (§§ 192, 196, 236)



11. Then $\dfrac{AF}{AB} = \dfrac{MF}{AF}$,

or $\overline{AF}^2 = AB \times MF$. (?)

12. Then by (1), $\dfrac{p'^2}{4\,n^2} = \dfrac{p}{n} \times \dfrac{P'}{4\,n} = \dfrac{p \times P'}{4\,n^2}$.

13. Then, $p'^2 = p \times P'$, and $p' = \sqrt{p \times P'}$. (3)

341. We will now show how to compute an approximate value of π (§ 333).

If the diameter of a ⊙ is 1, the side of an inscribed square is $\frac{1}{2}\sqrt{2}$ (§ 319); hence, its perimeter is $2\sqrt{2}$.

Again, the side of a circumscribed square is equal to the diameter of the ⊙; hence, its perimeter is 4.

We then put in equation (2), § 340,

$$P = 4, \text{ and } p = 2\sqrt{2} = 2.82843.$$

Then, $$P' = \frac{2\,P \times p}{P+p} = 3.31371.$$

We then put in equation (3), § 340,

$$p = 2.82843, \text{ and } P' = 3.31371.$$

Then, $$p' = \sqrt{p \times P'} = 3.06147.$$

These are the perimeters of the regular circumscribed and inscribed octagons, respectively.

Repeating the operation with these values, we put in (2),

$$P = 3.31371, \text{ and } p = 3.06147.$$

Then, $$P' = \frac{2\,P \times p}{P+p} = 3.18260.$$

We then put in (3), $p = 3.06147$ and $P' = 3.18260$.

Then, $$p' = \sqrt{p \times P'} = 3.12145.$$

These are, respectively, the perimeters of the regular circumscribed and inscribed polygons of sixteen sides.

In this way, we form the following table:

No. of Sides	Perimeter of Reg. Circ. Polygon	Perimeter of Reg. Insc. Polygon
4	4.	2.82843
8	3.31371	3.06147
16	3.18260	3.12145
32	3.15172	3.13655
64	3.14412	3.14033
128	3.14222	3.14128
256	3.14175	3.14151
512	3.14163	3.14157

The last result shows that the circumference of a ⊙ whose diameter is 1 is > 3.14157, and < 3.14163.

Hence, an approximate value of π is 3.1416, correct to the fourth decimal place.

Note. The value of π to fourteen decimal places is 3.14159265358979.

Ex. 29. The radius of a circle is 12. What is the radius of a circle having twice the area?

Ex. 30. The area of the equilateral triangle inscribed in a circle is one-half the area of the regular hexagon inscribed in the same circle.

Ex. 31. Two circumferences are in the ratio 3 to 4. What is the ratio of their radii? their diameters? their areas?

Ex. 32. The areas of two circles are in the ratio 1 to 2. What is the ratio of their radii? of their diameters?

Ex. 33. The areas of two circles are 836 and 616, respectively. If the radius of the second is 257, what is the radius of the first?

Ex. 34. One side of an inscribed equilateral triangle is 10; find the radius of the circle.

Ex. 35. What is the area of a sector of a circle whose radius is 12, if the angle of the sector is 45°?

Ex. 36. Find the ratio of the area of a circle to the area of its circumscribed square.

Ex. 37. Find the ratio of the area of a circle to the area of its circumscribed equilateral triangle.

Ex. 38. Find the ratio of the perimeters of the circumscribed and inscribed equilateral triangles.

Ex. 39. A wheel revolves 55 times in travelling $\frac{1045\pi}{4}$ ft. What is its diameter in inches?

If r represents the radius, a the apothem, s the side, and k the area, prove that

Ex. 40. In a regular octagon,

$$s = r\sqrt{2-\sqrt{2}},\ a = \tfrac{1}{2}r\sqrt{2+\sqrt{2}},\ \text{and}\ k = 2r^2\sqrt{2}.$$

Ex. 41. In a regular dodecagon,

$$s = r\sqrt{2-\sqrt{3}},\ a = \tfrac{1}{2}r\sqrt{2+\sqrt{3}},\ \text{and}\ k = 3r^2.$$

Ex. 42. In a regular octagon,

$$s = 2a(\sqrt{2}-1),\ r = a\sqrt{4-2\sqrt{2}},\ \text{and}\ k = 8a^2(\sqrt{2}-1).$$

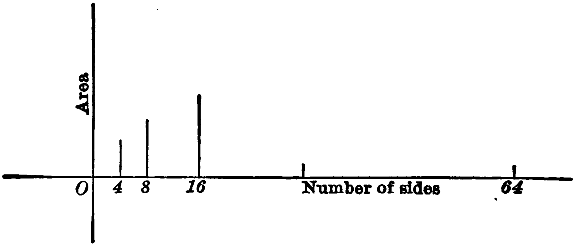
Ex. 43. In a regular dodecagon,

$$s = 2a(2-\sqrt{3}),\ \ r = 2a\sqrt{2-\sqrt{3}},\ \text{and}\ k = 12a^2(2-\sqrt{3}).$$

Ex. 44. In a regular decagon, $a = \frac{1}{4}r\sqrt{10+2\sqrt{5}}$.
(Find the apothem.)

Ex. 45. Find the number of degrees in a radian, an arc whose length is equal to that of the radius of the circle. ($\pi = 3.1416$.)

Ex. 46. A circular grass plot whose diameter is 52 feet is surrounded by a walk of uniform width whose area is 753.984. Find the width of the walk. ($\pi = 3.1416$.)



Ex. 47. Find the areas of regular inscribed polygons of 4, 8, 16, 32, 64, 128 sides, and of the circumscribing circle (§ 341). Using *area* for vertical measurements and *number of sides* for horizontal measurements, make a graph of the data showing the approach of the area of the polygon to that of the circle as the number of sides increases.

LOCI

342. Def. If a series of points, all of which satisfy a certain condition, lie in a certain line, and every point in this line satisfies the given condition, the line is said to be the *locus* of the points.

For example, all points which satisfy the condition of being equally distant from the extremities of a straight line, lie in the perpendicular erected at the middle point of the line (§ 56).

Also, every point in the perpendicular erected at the middle point of a line satisfies the condition of being equally distant from the extremities of the line (§ 55).

Hence, *the perpendicular erected at the middle point of a straight line is the* **Locus** *of points which are equally distant from the extremities of the line.*

Again, all points which satisfy the condition of being within an angle, and equally distant from its sides, lie in the bisector of the angle (§ 99).

Also, every point in the bisector of an angle satisfies the condition of being equally distant from its sides (§ 98).

Hence, *the bisector of an angle is the locus of points which are within the angle, and equally distant from its sides.*

Ex. 48. Find the locus of points in a plane three inches from a fixed point in the plane. Construct the locus.

Ex. 49. What is the locus of points in a plane equally distant from all points in the circumference of a circle?

Ex. 50. Find the locus of the vertices of triangles having a fixed base and the same altitude. Construct the locus.

Ex. 51. Two equal chords intersect; find the locus of their intersections.

Ex. 52. The length of the hypotenuse of a right triangle is fixed; find the locus of the centres of the inscribed circles; of the circumscribed circles.

Ex. 53. Find the locus of the middle points of chords drawn through a fixed point within a circle.

Ex. 54. Construct the locus of points equidistant from two given lines. How many cases are there?

Ex. 55. Draw the approximate locus of the middle points of the hypotenuses of all right triangles having a hypotenuse of fixed length and the vertex of the right angle fixed.

Ex. 56. Construct the locus of the centres of circles tangent to two lines which intersect at right angles. Is there more than one system of these circles?

Ex. 57. Find the locus of the centres of circumferences which pass through a fixed point. 213